



The CENTRE for EDUCATION  
in MATHEMATICS and COMPUTING  
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## ***2023 Hypatia Contest***

**Wednesday, April 5, 2023**  
(in North America and South America)

**Thursday, April 6, 2023**  
(outside of North America and South America)

*Solutions*

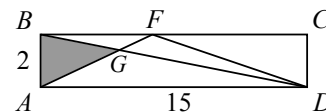
1. (a) Jasmin's total score was  $3 \cdot 2 + 4 \cdot 5 = 6 + 20$  or 26 points.
- (b) Suppose that Sam had  $n$  throws that each scored 5 points; then Sam had  $2n$  throws that each scored 2 points.  
 Since Sam's total score was 36 points, then  $5 \cdot n + 2 \cdot 2n = 36$  or  $9n = 36$ , and so  $n = 4$ .  
 In total, Sam took  $n + 2n = 3n$  throws, which is  $3 \cdot 4 = 12$  throws.
- (c) Since Tia's total score was 37 points, then  $2t + 5f = 37$ .  
 Since  $2t$  is even for all integer values of  $t$ , then  $5f$  must be odd since their sum is 37 (which is odd).  
 The value of  $5f$  is odd exactly when  $f$  is odd.  
 When  $f = 1$ , we get  $2t + 5 = 37$  or  $2t = 32$ , and so  $t = 16$ .  
 When  $f = 3$ , we get  $2t + 15 = 37$  or  $2t = 22$ , and so  $t = 11$ .  
 When  $f = 5$ , we get  $2t + 25 = 37$  or  $2t = 12$ , and so  $t = 6$ .  
 When  $f = 7$ , we get  $2t + 35 = 37$  or  $2t = 2$ , and so  $t = 1$ .  
 When  $f \geq 9$ ,  $5f \geq 45$  and so  $2t + 5f > 37$ .  
 The possible ordered pairs  $(t, f)$  are  $(16, 1)$ ,  $(11, 3)$ ,  $(6, 5)$ , and  $(1, 7)$ .
- (d) If  $a$  throws each score 6 points and  $b$  throws each score 21 points, then  $6a + 21b$  or  $3(2a + 7b)$  points are scored.  
 Since  $a$  and  $b$  are non-negative integers, then  $2a + 7b$  is a non-negative integer, and so the total number of points scored,  $3(2a + 7b)$ , is a multiple of 3.  
 Since 182 is not a multiple of 3, then it is not possible to have a total score of 182 points.

2. (a) The total area of the shaded regions can be determined by subtracting the area of  $\triangle AED$  from the area of rectangle  $ABCD$ .  
 The area of rectangle  $ABCD$  is  $2 \cdot 15 = 30$ .  
 The base length of  $\triangle AED$  is  $AD = 15$ , its height is equal to  $AB = 2$ , and so its area is  $\frac{1}{2} \cdot 15 \cdot 2 = 15$ .  
 Thus, the total area of the shaded regions is  $30 - 15 = 15$ .

(b) *Solution 1*

Since  $\triangle AFD$  has base length  $AD = 15$  and height equal to  $AB = 2$ , then its area is  $\frac{1}{2} \cdot 15 \cdot 2 = 15$ .

The area of  $\triangle AFD$  is equal to the sum of the areas of  $\triangle AGD$  and  $\triangle FGD$ .



Since the area of  $\triangle AFD$  is 15, and the area of  $\triangle FGD$  is 5, then the area of  $\triangle AGD$  is  $15 - 5 = 10$ .

The area of  $\triangle ABD$  is  $\frac{1}{2} \cdot 15 \cdot 2 = 15$ .

The area of  $\triangle ABD$  is equal to the sum of the areas of  $\triangle AGD$  and  $\triangle ABG$ .

Since the area of  $\triangle ABD$  is 15, and the area of  $\triangle AGD$  is 10, then the area of  $\triangle ABG$  (the shaded area) is  $15 - 10 = 5$ .

*Solution 2*

Consider  $\triangle AFD$  and  $\triangle ABD$ . Each has base  $AD$  and height  $AB$  and thus the two triangles have equal areas.

The area of  $\triangle AFD$  is equal to the sum of the areas of  $\triangle AGD$  and  $\triangle FGD$ .

The area of  $\triangle ABD$  is equal to the sum of the areas of  $\triangle AGD$  and  $\triangle ABG$ .

Thus, the area of  $\triangle ABG$  (the shaded area) is equal to the area of  $\triangle FGD$ , which is 5.

- (c) The total area of the bottom two shaded regions ( $\triangle ASR$  and  $\triangle RQD$ ) can be determined by subtracting the area of  $PQRS$  from the area of  $\triangle APD$ .

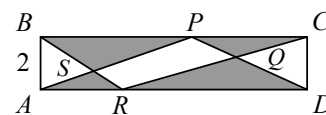
The base length of  $\triangle APD$  is  $AD = 15$ , its height is equal to  $AB = 2$ , and so its area is  $\frac{1}{2} \cdot 15 \cdot 2 = 15$ .

Since the area of  $PQRS$  is 6, the total area of the two bottom shaded regions is  $15 - 6 = 9$ .

We can similarly show that the total area of the two top shaded regions ( $\triangle BSP$  and  $\triangle PQC$ ) can be determined by subtracting the area of  $PQRS$  from the area of  $\triangle BRC$ .

Since  $\triangle BRC$  also has area 15, then the total area of the two top shaded regions is also  $15 - 6 = 9$ .

Thus, the total area of the shaded regions is  $9 + 9 = 18$ .



3. (a) Switching the second and fourth digits of 6238 gives the cousin 6832.

Since 6832 is not in the list, then it is the missing cousin.

- (b) Of the 16 six-digit integers in the given list, 15 are cousins of the original integer.

If the original six-digit integer is  $abcdef$ , then there are exactly 5 cousins for which  $a$  is not the 1st digit.

These 5 cousins are a result of switching the 1st digit,  $a$ , with each of the other 5 digits,  $b, c, d, e, f$ , and thus are  $bacdef, cbadef, dbcaef, ebcdafe,$  and  $fbcdea$ .

Each of the remaining  $15 - 5 = 10$  cousins of  $abcdef$  have first digit  $a$ .

Since the digits are distinct, then the 1st digit of the original integer is the integer that occurs most frequently as the 1st digit in the given list, which is 7.

A similar argument can be made for each of the other digits.

That is, the  $n$ th digit of the original integer is the integer that occurs most frequently as the  $n$ th digit in the given list.

Thus, the original integer has 2nd digit 2, 3rd digit 6, 4th digit 4, 5th digit 9, 6th digit 1, and so the original integer is 726491.

We may check that if the original integer is 726491, then the cousins are indeed the remaining 15 integers in the given list.

- (c) The cousins of the three-digit integer  $cd3$  are  $dc3, 3dc$  and  $c3d$ .

The difference between the two integers  $cd3$  and  $dc3$  could be negative, in which case it's not the right difference. If it is positive, then  $cd3$  minus  $dc3$  has ones digit 0, and thus cannot equal  $d95$ .

Similarly, the difference between the two integers  $cd3$  and  $c3d$  could be negative, in which case it's not the right difference. Since  $cd3$  and  $c3d$  each have hundreds digit  $c$ , then  $cd3$  minus  $c3d$  has hundreds digit 0, and thus cannot equal  $d95$  (since  $d$  is a non-zero digit).

Therefore,  $cd3$  minus  $3dc$  is equal to  $d95$ .

Since the difference,  $d95$ , has ones digit 5, then  $c = 8$ . (You should confirm for yourself that this is the only possible value for  $c$  before moving on.)

With  $c = 8$ , we get  $8d3$  minus  $3d8$  is equal to  $d95$ .

The three-digit integer  $8d3$  is equal to  $800 + 10d + 3$ .

The three-digit integer  $3d8$  is equal to  $300 + 10d + 8$ .

Thus,  $8d3$  minus  $3d8$  is equal to  $(800 + 10d + 3) - (300 + 10d + 8) = 500 - 5 = 495$ , which is equal to  $d95$ , and so  $d = 4$ .

(We may confirm that  $843 - 348 = 495$ .)

The value of  $c$  is 8 and the value of  $d$  is 4, and no other values are possible.

(d) *Solution 1*

The six cousins of the four-digit integer  $mn97$  are  $nm97$ ,  $9nm7$ ,  $7n9m$ ,  $m9n7$ ,  $m79n$ , and  $mn79$ .

The cousin  $nm97$  is equal to  $1000n + 100m + 90 + 7$ .

The cousin  $9nm7$  is equal to  $9000 + 100n + 10m + 7$ .

The cousin  $7n9m$  is equal to  $7000 + 100n + 90 + m$ .

The cousin  $m9n7$  is equal to  $1000m + 900 + 10n + 7$ .

The cousin  $m79n$  is equal to  $1000m + 700 + 90 + n$ .

The cousin  $mn79$  is equal to  $1000m + 100n + 70 + 9$ .

The sum,  $S$ , of the six cousins is thus

$$\begin{aligned} S &= 1000(n + 9 + 7 + m + m + m) + 100(m + n + n + 9 + 7 + n) + 10(9 + m + 9 + n + 9 + 7) \\ &\quad + (7 + 7 + m + 7 + n + 9) \\ &= 1000(3m + n + 16) + 100(m + 3n + 16) + 10(m + n + 34) + (m + n + 30) \end{aligned}$$

The sum,  $S$ , must be equal to the five-digit integer  $nmnm7$ , which has ones digit 7.

The ones digit of  $S = 1000(3m + n + 16) + 100(m + 3n + 16) + 10(m + n + 34) + (m + n + 30)$  is equal to the ones digit of  $m + n + 30$ , which is equal to the ones digit of  $m + n$  (since the ones digit of 30 is 0).

If the ones digit of  $m + n$  is 7, then  $m + n = 7$  or  $m + n = 17$ .

Since  $m$  and  $n$  are distinct non-zero digits, then the possible pairs  $(m, n)$  are  $(1, 6)$ ,  $(2, 5)$ ,  $(3, 4)$ ,  $(4, 3)$ ,  $(5, 2)$ ,  $(6, 1)$ ,  $(8, 9)$ , and  $(9, 8)$ .

If  $(m, n) = (1, 6)$ , then the five-digit integer  $nmnm7$  is 61617, and

$$\begin{aligned} S &= 1000(3m + n + 16) + 100(m + 3n + 16) + 10(m + n + 34) + (m + n + 30) \\ &= 1000(3(1) + 6 + 16) + 100(1 + 3(6) + 16) + 10(1 + 6 + 34) + (1 + 6 + 30) \\ &= 1000(25) + 100(35) + 10(41) + 37 \\ &= 28947 \end{aligned}$$

and so  $(m, n) = (1, 6)$  is not a possible pair.

We continue to check the remaining possible pairs  $(m, n)$  in the table below.

$(m, n)$	$nmnm7$	$S$
$(2, 5)$	52527	30747
$(3, 4)$	43437	32547
$(4, 3)$	34347	34347
$(5, 2)$	25257	36147
$(6, 1)$	16167	37947
$(8, 9)$	98987	54657
$(9, 8)$	89897	56457

The only pair of distinct, non-zero digits  $(m, n)$  is  $(4, 3)$ .

*Solution 2*

The six cousins of the four-digit integer  $mn97$  are  $nm97$ ,  $9nm7$ ,  $7n9m$ ,  $m9n7$ ,  $m79n$ , and  $mn79$ . As in Solution 1, the sum,  $S$ , of the six cousins is

$$\begin{aligned} S &= 1000(n + 9 + 7 + m + m + m) + 100(m + n + n + 9 + 7 + n) + 10(9 + m + 9 + n + 9 + 7) \\ &\quad + (7 + 7 + m + 7 + n + 9) \\ &= 1000(3m + n + 16) + 100(m + 3n + 16) + 10(m + n + 34) + (m + n + 30) \\ &= 3000m + 1000n + 16000 + 100m + 300n + 1600 + 10m + 10n + 340 + m + n + 30 \\ &= 3111m + 1311n + 17970 \end{aligned}$$

The sum,  $S$ , must be equal to the five-digit integer  $nmnm7$ , which is equal to  $10\,000n + 1000m + 100n + 10m + 7$  or  $10\,100n + 1010m + 7$ .

Setting these two expressions equal to one another and simplifying, we get

$$\begin{aligned} 3111m + 1311n + 17970 &= 10\,100n + 1010m + 7 \\ 8789n - 2101m &= 17963 \\ 799n - 191m &= 1633 \quad (\text{after division by } 11) \end{aligned}$$

Since  $799n = 1633 + 191m$  and  $m \geq 1$ , then  $799n \geq 1633 + 191(1)$  or  $799n \geq 1824$ , and so  $n \geq \frac{1824}{799} \approx 2.3$ .

Since  $m \leq 9$ , then  $799n \leq 1633 + 191(9)$  or  $799n \leq 3352$ , and so  $n \leq \frac{3352}{799} \approx 4.2$ .

Since  $2.3 \leq n \leq 4.2$  and  $n$  is an integer, then the possible values of  $n$  are 3 and 4.

Substituting  $n = 4$ , we get  $799(4) = 1633 + 191m$  or  $191m = 1563$ , and since  $\frac{1563}{191}$  is not an integer, then  $n \neq 4$ .

Substituting  $n = 3$ , we get  $799(3) = 1633 + 191m$  or  $191m = 764$ , and so  $m = 4$ .

When  $m = 4$  and  $n = 3$ , the five-digit integer  $nmnm7$  is 34347 and  $S = 3111(4) + 1311(3) + 17970$  or  $S = 34347$ , and so the only pair of distinct, non-zero digits  $(m, n)$  is  $(4, 3)$ .

4. (a) Since there are 9 ways to generate each of the three integers, there are a total of  $9 \cdot 9 \cdot 9 = 729$  ways to generate all three integers.

Next, we consider the different cases for which the product of the three integers is a prime number.

The integers from 1 to 9 include the composite numbers 4, 6, 8, and 9.

If any one of the three integers generated is composite, then the product of the three integers is composite.

Thus, the integers must be chosen from 1, 2, 3, 5, and 7.

Of these five integers, 2, 3, 5, and 7 are prime numbers.

If two or more of the three integers generated are prime numbers, then the product of the three integers is composite.

Thus, at most one integer is a prime number.

If no integer generated is a prime number (all three are equal to 1), then the product is 1, which is not prime.

Thus, Amarpreet's product is a prime number exactly when one of the integers is 2, 3, 5, or 7, and each of the other two integers is equal to 1.

There are 4 ways to choose one of the prime numbers,  $p$ , and 3 ways to arrange the integers 1, 1,  $p$ , and so there are  $4 \cdot 3 = 12$  ways to generate three integers whose product is a prime number.

Therefore, the probability that the product is a prime number is  $\frac{12}{729} = \frac{4}{243}$ .

- (b) *Solution 1*

We begin by considering the different cases for which the product of the four integers is divisible by 5, but not divisible by 7.

The only integer from 1 to 9 that is divisible by 5 is 5, and so the product of the four integers is divisible by 5 exactly when at least one of the four integers is a 5.

Similarly, the only integer from 1 to 9 that is divisible by 7 is 7, and so the product of the four integers is not divisible by 7 exactly when a 7 is not among the four integers

generated.

We proceed to count the number of ways to generate such arrangements of four integers by considering the number of times that 5 appears in the arrangement.

Case 1: there are four 5s

Since each of the four integers must be 5, there is 1 such possibility in this case.

Case 2: there are exactly three 5s

The three 5s can be arranged in 4 different ways: 555\_, 55\_5, 5\_55, \_555.

There are 7 choices for the integer that is not a 5 since it can be any of the nine integers, except 5 and 7.

Thus, there are  $4 \cdot 7 = 28$  possibilities in this case.

Case 3: there are exactly two 5s

The two 5s can be arranged in 6 different ways: 55\_\_, 5\_5\_, 5\_\_5, \_55\_, \_5\_5, \_\_55.

Once the two 5s have been placed, there are 7 choices for each of the remaining two integers (since it is possible that these integers are equal to one another).

Thus, there are  $6 \cdot 7 \cdot 7 = 294$  possibilities in this case.

Case 4: there is exactly one 5

The 5 can be placed in 4 different ways.

Once the 5 has been placed, there are 7 choices for each of the remaining three integers (since it is possible that these integers are equal to one another).

Thus, there are  $4 \cdot 7 \cdot 7 \cdot 7 = 1372$  possibilities in this case.

In total, there are  $1 + 28 + 294 + 1372 = 1695$  ways to select the four integers.

Since there are 9 ways to generate each of the four integers, there are a total of  $9^4 = 6561$  ways to generate all four integers.

Therefore, the probability that Braxton's product is divisible by 5, but not divisible by 7 is  $\frac{1695}{6561} = \frac{565}{2187}$ .

### *Solution 2*

Let  $A$  be the event that the product generated by Braxton is divisible by 5, and  $\bar{A}$  be the event that the product is *not* divisible by 5.

Let  $B$  be the event that the product generated by Braxton is divisible by 7, and  $\bar{B}$  be the event that the product is *not* divisible by 7.

We are asked to find the probability of  $A$  and  $\bar{B}$ , which we write as  $P(A \text{ and } \bar{B})$ .

Consider the Venn diagram shown in Figure 1.

The shaded region of this diagram represents  $P(A \text{ and } \bar{B})$ .

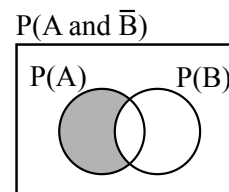


Figure 1

We wish to determine a more efficient method for determining  $P(A \text{ and } \bar{B})$  than that which was shown in Solution 1.

To do so, we proceed to use Venn diagrams to help express  $P(A \text{ and } \bar{B})$  in an equivalent form.

Consider the next two Venn diagrams shown below.

In Figure 2, the shaded region represents  $P(\overline{B})$ .

In Figure 3, the shaded region represents  $P(\overline{A \text{ and } \overline{B}})$ .

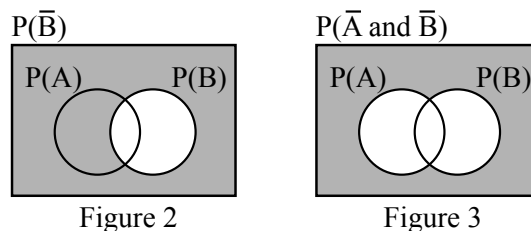


Figure 2

Figure 3

Notice that if the region that is shaded in Figure 3 is removed (unshaded) in Figure 2, then the resulting Venn diagram is equivalent to that in Figure 1.

That is, mathematically,  $P(A \text{ and } \overline{B}) = P(\overline{B}) - P(\overline{A \text{ and } \overline{B}})$ .

(Without the use of the Venn diagrams, we may have similarly noticed that since exactly one of  $A$  and  $\overline{A}$  occurs, then  $P(\overline{B}) = P(A \text{ and } \overline{B}) + P(\overline{A} \text{ and } \overline{B})$ , and so  $P(\overline{B}) - P(\overline{A} \text{ and } \overline{B}) = P(A \text{ and } \overline{B}) + P(\overline{A} \text{ and } \overline{B}) - P(\overline{A} \text{ and } \overline{B}) = P(A \text{ and } \overline{B})$ .)

Thus, the probability that the product is divisible by 5, but not divisible by 7, is equal to the probability that the product is not divisible by 7, minus the probability that the product is both not divisible by 5 and not divisible by 7.

The only integer from 1 to 9 that is divisible by 7 is 7, and so the product of the four integers is not divisible by 7 exactly when a 7 is not among the four integers generated.

The probability that the integer generated is not 7 is  $\frac{8}{9}$ , and so the probability that the

product of the four integers is not divisible by 7 is  $P(\overline{B}) = \left(\frac{8}{9}\right)^4$ .

The only integer from 1 to 9 that is divisible by 5 is 5, and so the product of the four integers is not divisible by 5 exactly when a 5 is not among the four integers generated.

The probability that the integer generated is not divisible by both 5 and 7 is thus  $\frac{7}{9}$ , and

so the probability that the product of the four integers is not divisible 5 and not divisible by 7 is  $P(\overline{A} \text{ and } \overline{B}) = \left(\frac{7}{9}\right)^4$ .

Therefore, the probability that the product is divisible by 5, but not divisible by 7, is

$$\left(\frac{8}{9}\right)^4 - \left(\frac{7}{9}\right)^4 = \frac{8^4 - 7^4}{9^4} = \frac{4096 - 2401}{6561} = \frac{1695}{6561} = \frac{565}{2187}.$$

### Solution 3

Suppose that the four numbers generated are  $a, b, c, d$ .

When there are no restrictions, there are 9 choices for each of  $a, b, c, d$ , and so a total of  $9^4$  possible combinations of 4 digits that can be generated.

If  $abcd$  is not divisible by 7, then none of  $a, b, c, d$  can be 7; thus, there are 8 possibilities for each of  $a, b, c, d$ , and so  $8^4$  of these  $9^4$  combinations give products that are not divisible by 7.

If  $abcd$  is not divisible by 7 and not divisible by 5, then none of  $a, b, c, d$  can be 5; thus, there are 7 possibilities for each of  $a, b, c, d$ , and so  $7^4$  of these  $8^4$  combinations whose products are not divisible by 7 give products that are also not divisible by 5.

This means that there are  $8^4 - 7^4$  combinations whose product is not divisible by 7 but

whose product is divisible by 5.

Therefore, the probability that a random combination gives a product that is not divisible by 7 but is divisible by 5 is  $\frac{8^4 - 7^4}{9^4} = \frac{4096 - 2401}{6561} = \frac{1695}{6561} = \frac{565}{2187}$ .

(c) If any one of the 2023 integers generated is a 6, then the product is divisible by 6.

Thus, we exclude 6 from the choice of possible integers.

Further, an integer is divisible by 6 exactly when it is divisible by both 2 and 3.

With 6 excluded, the remaining integers that are divisible by 2 are 2, 4 and 8, and the remaining integers that are divisible by 3 are 3 and 9.

Therefore, the product is also divisible by 6 when at least one of the integers generated is 2 or 4 or 8, and at least one of the integers generated is 3 or 9.

To summarize, the product is not divisible by 6 exactly when

- 6 is not among the integers generated, and
- there are integers from at most one of the two lists 2, 4, 8, and 3, 9.

Thus, there are exactly 3 cases which produce a product that is not divisible by 6, as follows.

Case	Is there a 6?	Is there a 2, 4 or 8?	Is there a 3 or 9?
1	no	no	no
2	no	no	yes
3	no	yes	no

For each of these 3 cases, we count the number of ways that the product generated is not divisible by 6.

Case 1: there is no 6, 2, 4, 8, 3, or 9

In this case, there are 3 integers that may be chosen (1, 5 and 7), and so there are  $3^{2023}$  ways to generate this product.

Case 2: there is no 6, 2, 4, or 8

In this case, there are 5 integers that may be chosen (1, 3, 5, 7, and 9), and so there are  $5^{2023}$  ways to generate this product.

However, this count includes the cases in which only 1, 5 and 7 are chosen (3 and 9 are not), which were counted in Case 1.

Thus, the number of ways to generate the products in Case 2, different from those in Case 1, is  $5^{2023} - 3^{2023}$ .

Case 3: there is no 6, 3, or 9

In this case, there are 6 integers that may be chosen (1, 2, 4, 5, 7, and 8), and there are  $6^{2023}$  ways to generate this product.

However, this count includes the cases in which only 1, 5 and 7 are chosen (2, 4 and 8 are not), which were counted in Case 1.

Thus, the number of ways to generate the products in Case 3, different from those in Case 1, is  $6^{2023} - 3^{2023}$ .

Thus, the total number of ways to generate all products that are not divisible by 6 is

$$3^{2023} + 5^{2023} - 3^{2023} + 6^{2023} - 3^{2023} = 5^{2023} + 6^{2023} - 3^{2023}$$

When there are no restrictions, there are 9 choices for each integer generated, and so there is a total of  $9^{2023}$  possible ways to generate the product.



Thus, the probability,  $p$ , that the product is not divisible by 6 is

$$p = \frac{5^{2023} + 6^{2023} - 3^{2023}}{9^{2023}}$$

and so  $p \cdot 9^{2023} = 5^{2023} + 6^{2023} - 3^{2023}$ , which is equal to the sum and difference of integers and thus an integer.

To determine the ones digit of the integer equal to  $p \cdot 9^{2023}$ , we determine the ones digit of the integer equal to  $5^{2023} + 6^{2023} - 3^{2023}$ .

The ones digit of  $5^a$  is equal to 5 for all positive integers  $a$ , and thus the ones digit of  $5^{2023}$  is 5.

The ones digit of  $6^b$  is equal to 6 for all positive integers  $b$ , and thus the ones digit of  $6^{2023}$  is 6.

Consider the ones digit of successive integer powers of 3 beginning at  $3^1$ :

- $3^1$  has ones digit 3
- $3^2$  has ones digit 9
- $3^3$  has ones digit 7
- $3^4$  has ones digit 1
- $3^5$  has ones digit 3

Since  $3^1$  and  $3^5$  have the same ones digit and we are performing the same action to get from one step to the next, the results begin to repeat. Thus the block of 4 ones digits (3, 9, 7, 1) must form a cycle.

Since  $2023 = 4 \cdot 505 + 3$ , then the ones digit of  $3^{2023}$  is the third number in the repeating block, which is 7.

Finally, the ones digit of the integer equal to  $p \cdot 9^{2023}$  is the ones digit of  $5 + 6 - 7$ , which is equal to 4.