The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2023 Fryer Contest

Wednesday, April 5, 2023
(in North America and South America)

Thursday, April 6, 2023
(outside of North America and South America)

Solutions

1. (a) Lonnie rests for 30 s between the 1st and 2nd sprints, the 2nd and 3rd sprints, and so on up to and including the 23rd and 24th sprints.
Thus, Lonnie takes the 30 s rest 23 times.

(b) Since Lonnie sprints at a constant speed of 8 m/s, then it takes Lonnie $\dfrac{200 \text{ m}}{8 \text{ m/s}} = 25$ s to sprint 200 m.
Lonnie completes 24 such sprints, and so his time spent sprinting is $24 \times 25 \text{ s} = 600$ s.
Lonnie also takes 23 rests, each of length 30 s, and so his time spent resting is $23 \times 30 \text{ s} = 690$ s.
On Monday, Lonnie's total practice time is thus $600 \text{ s} + 690 \text{ s} = 1290$ s.

(c) *Solution 1*
On Tuesday, each of Lonnie's 240 m sprints takes $\dfrac{240 \text{ m}}{8 \text{ m/s}} = 30$ s, and so Lonnie spends $20 \times 30 \text{ s} = 600$ s sprinting.
Lonnie rests 19 times, and so he rests for a total of $19 \times 30 \text{ s} = 570$ s.
On Tuesday, Lonnie's total practice time is thus $600 \text{ s} + 570 \text{ s} = 1170$ s, and so Tuesday's practice takes $1290 - 1170 = 120$ fewer seconds compared to Monday's practice.

Solution 2
On Monday, Lonnie sprints $24 \times 200 \text{ m} = 4800$ m.
On Tuesday, Lonnie also sprints $20 \times 240 \text{ m} = 4800$ m.
Since Lonnie sprints at the same constant speed on both days, then he spends the same amount of time sprinting on each of the two days.
Thus, the difference between the length of time that Lonnie practices on the two days is the difference between the time that he spends resting between sprints.
Lonnie rests 23 times on Monday, and he rests 19 times on Tuesday.
Since he rests 4 more times on Monday than he does on Tuesday, then Tuesday's practice takes $4 \times 30 = 120$ fewer seconds compared to Monday's practice.

2. (a) The 5th row includes the integers 17, 19, 21, 23, and 25, and so the average of the integers in the 5th row is $\dfrac{17+19+21+23+25}{5} = 21$.

(b) The row that has the integer 145 in the 1st position must immediately follow the row that has 144 in the last position.
Since $12^2 = 144$, then the integer 144 is in the last position of the 12th row, and so 145 is in the first position of the 13th row.

(c) Since $40^2 = 1600$, then the integer in the last position (the 40th position) of the 40th row is 1600.
When moving from right to left along each row, the integers decrease by 2, and so the integer in the 39th position of the 40th row is $1600 - 2 = 1598$.

(d) *Solution 1*
Moving from left to right along a row, the integers increase by a constant (namely 2), and so the average of the integers in a row is equal to the average of the integer in the first position of the row and the integer in the last position of the row. Can you see why this is true?
Since $15^2 = 225$, then the integer in the last position of the 15th row is 225, and so the integer in the first position of the 16th row is 226.
Since $16^2 = 256$, then the integer in the last position of the 16th row is 256.

Thus, the average of the integers in the 16th row is $\dfrac{226+256}{2} = 241$, and so $r = 16$.

Solution 2

Since $15^2 = 225$, then each of the entries in the first 15 rows is at most 225.

This means that the average of the entries in each row up to and including the 15th row must be at most 225.

Since $16^2 = 256$, then each of the entries in the rows after the 16th row is greater than 256. This means that the average of the entries in each row after the 16th must be greater than 256.

This means r must be greater than 15 and must be smaller than 17. In other words, $r = 16$.

We can check that the entries in row 16 are

$$226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248, 250, 252, 254, 256$$

and the average of these integers is indeed 241.

3. (a) The five-digit positive integer $4B5B2$ is divisible by 3 exactly when the sum of its digits, $4 + B + 5 + B + 2 = 2B + 11$ is divisible by 3.

Checking the possible values of B,

Value of B	1	2	3	4	5	6	7	8	9
Value of $2B + 11$	13	15	17	19	21	23	25	27	29

we get that $2B + 11$ is divisible by 3 when $B = 2$, $B = 5$ and $B = 8$.

(b) *Solution 1*

Since $ABABA$ is not divisible by 3, then $A + B + A + B + A = 3A + 2B$ is not divisible by 3.

Since $3A$ is divisible by 3 for all possible values of the digit A, then if $2B$ were also divisible by 3 (that is, if $B = 3$, 6, 9), it would be the case that $3A + 2B$ is divisible by 3.

Since $3A + 2B$ is not divisible by 3, then $2B$ cannot be divisible by 3 and so the possible values of B are 1, 2, 4, 5, 7, 8.

Since $ABABA$ is divisible by 4, then the two-digit positive integer BA is divisible by 4.

For each of the possible values of B, namely $B = 1$, 2, 4, 5, 7, 8, we determine the values of A for which BA is divisible by 4 .

For example when $B = 1$, the two-digit positive integer $1A$ is divisible by 4 exactly when $A = 2$ or $A = 6$.

In the table below, we determine the remaining pairs A and B that are possible.

B	A	(A, B)
1	2, 6	$(2, 1)$, $(6, 1)$
2	4, 8	$(4, 2)$, $(8, 2)$
4	4, 8	$(4, 4)$, $(8, 4)$
5	2, 6	$(2, 5)$, $(6, 5)$
7	2, 6	$(2, 7)$, $(6, 7)$
8	4, 8	$(4, 8)$, $(8, 8)$

Thus, there are 12 different pairs of non-zero digits A and B that are possible.

Solution 2
Since $ABABA$ is divisible by 4, then it is also divisible by 2 and thus even.
Since $ABABA$ is even, then the ones digit is even, and so the possible values of A are 2, 4, 6, 8.
Since $ABABA$ is divisible by 4, then the two-digit positive integer BA is divisible by 4.
For each of the possible values of A, namely $A = 2$, 4, 6, 8, we determine the values of B for which BA is divisible by 4 .
For example when $A = 2$, the two-digit positive integer $B2$ is divisible by 4 exactly when $B = 1$, 3, 5, 7, and 9.
When $A = 4$, the two-digit positive integer $B4$ is divisible by 4 exactly when $B = 2$, 4, 6, and 8.
When $A = 6$, the two-digit positive integer $B6$ is divisible by 4 exactly when $B = 1$, 3, 5, 7, and 9.
When $A = 8$, the two-digit positive integer $B8$ is divisible by 4 exactly when $B = 2$, 4, 6, and 8.
Finally, we consider the fact that $ABABA$ is not divisible by 3.
As was shown in Solution 1, the possible values of B are 1, 2, 4, 5, 7, 8 ($B \neq 3, 6, 9$).
Combining this information with the previous values of A and B, we get that the possible pairs of non-zero digits (A, B) are $(2, 1)$, $(2, 5)$, $(2, 7)$, $(4, 2)$, $(4, 4)$, $(4, 8)$, $(6, 1)$, $(6, 5)$, $(6, 7)$, $(8, 2)$, $(8, 4)$, and $(8, 8)$.
Thus, there are 12 different pairs of non-zero digits A and B that are possible.

(c) If $t = ACA2 \times BAC$ is divisible by 15, then t is divisible by both 5 and 3.
An integer is divisible by 5 exactly when its ones digit is 0 or 5, and so $ACA2$ is not divisible by 5.
Since $t = ACA2 \times BAC$ is divisible by 5 and $ACA2$ is not, then BAC must be divisible by 5, which means that $C = 5$ (since C is a non-zero digit).
Substituting $C = 5$, we get $t = A5A2 \times BA5$.

Since t is divisible by 3 and 3 is a prime number, then at least one of $A5A2$ or $BA5$ is divisible by 3, and so $A + 5 + A + 2 = 2A + 7$ is divisible by 3, or $B + A + 5$ is divisible by 3, or both are divisible by 3.

Since t is not divisible by 12, but t is divisible by 3, then t is not divisible by 4.
The three-digit integer $BA5$ is not divisible by 2 (and thus not divisible by 4) for all possible values of A, and so the four-digit integer $A5A2$ must not be divisible by 4 .
The four-digit integer $A5A2$ is divisible by 4 when the two-digit integer $A2$ is divisible by 4, or when $A = 1$, 3, 5, 7, or 9, and so the possible values of A are 2, 4, 6, 8.

Finally, we return to the requirement that $t = A5A2 \times BA5$ is divisible by 3, meaning that at least one of $2A + 7$ or $B + A + 5$ is divisible by 3.

When $A = 2$, $2A + 7 = 11$ which is not divisible by 3 and so $B + A + 5 = B + 7$ must be divisible by 3. $B + 7$ is divisible by 3 exactly when $B = 2, 5$, or 8, and so there are 3 triples A, B, C that are possible in this case.

When $A = 4$, $2A + 7 = 15$ which is divisible by 3, which means that B can be equal to any non-zero digit, and so there are 9 triples A, B, C that are possible in this case.

When $A = 6$, $2A + 7 = 19$ which is not divisible by 3 and so $B + A + 5 = B + 11$ must be divisible by 3. $B + 11$ is divisible by 3 exactly when $B = 1$, 4, or 7, and so there are 3 triples A, B, C that are possible in this case.

When $A = 8$, $2A + 7 = 23$ which is not divisible by 3 and so $B + A + 5 = B + 13$ must be divisible by 3. $B + 13$ is divisible by 3 exactly when $B = 2$, 5 or 8, and so there are 3 triples A, B, C that are possible in this case.

Therefore, there are $3+9+3+3 = 18$ different triples of non-zero digits A, B, C that are possible.

4. (a) Computer 1 is an odd-numbered computer, and so each cord connecting Computer 1 to another odd-numbered computer is red.
Thus, there is a route from Computer 1 to each of the odd-numbered computers from 3 to 49, inclusive, that uses only red cords.
Each cord between an odd-numbered computer and an even-numbered computer is blue.
Computer 1 is an odd-numbered computer and so every possible route from Computer 1 to an even-numbered computer must use at least one blue cord.
Thus there is no route from Computer 1 to any even-numbered computer that uses only red cords.
There are 24 odd numbers between 2 and 50, and so there are 24 possible values for n.

(b) Two integers have *different parity* if one integer is even and the other is odd.
Two integers have the *same parity* if they are both even or if they are both odd.
There are two cases to consider: A and B have different parity, or they have the same parity.
If A and B have different parity, then the cord between Computer A and Computer B is blue, and so there is a route between them that uses only blue cords.
If A and B have the same parity, then choose a number C that has different parity than that of A and B.
The cord between Computer A and Computer C is blue, and the cord between Computer C and Computer B is blue, and so the route from Computer A to Computer C to Computer B uses only blue cords.
Thus, for every pair of distinct computers, Computer A and Computer B, there is always a route between them that uses only blue cords.

(c) If the cord between Computer 13 and Computer 14 is yellow, then there is a route between them that uses only yellow cords, so assume that the cord between them is green.
Since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then the cord between Computer 1 and Computer 50 must be yellow.
Further, since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then at least one of the following must be true:

(i) the cord between Computer 1 and Computer 13 is yellow, or
(ii) the cord between Computer 13 and Computer 50 is yellow,

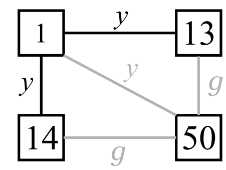
otherwise the route from Computer 1 to Computer 13 to Computer 50 uses only green cords.
Similarly, since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then at least one of the following must be true:

(iii) the cord between Computer 1 and Computer 14 is yellow, or
(iv) the cord between Computer 14 and Computer 50 is yellow,

otherwise the route from Computer 1 to Computer 14 to Computer 50 uses only green cords.
Since at least one of (i) or (ii) must be true, and at least one of (iii) or (iv) must be true, then there are 4 cases to consider, as follows.

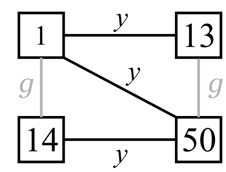
Case A: (i) and (iii) are true

In this case, the cord between Computer 1 and Computer 13 is yellow, and the cord between Computer 1 and Computer 14 is yellow, and so the route from Computer 13 to Computer 1 to Computer 14 uses only yellow cords.



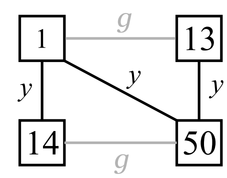
Case B: (i) and (iv) are true

In this case, the cord between Computer 1 and Computer 13 is yellow, and the cord between Computer 14 and Computer 50 is yellow.
Recall that the cord between Computer 1 and Computer 50 is also yellow, and so the route from Computer 13 to Computer 1 to Computer 50 to Computer 14 uses only yellow cords.



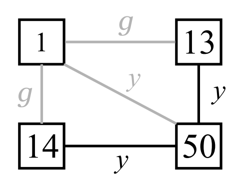
Case C: (ii) and (iii) are true

In this case, the cord between Computer 13 and Computer 50 is yellow, and the cord between Computer 1 and Computer 14 is yellow.
Since the cord between Computer 1 and Computer 50 is also yellow, then the route from Computer 13 to Computer 50 to Computer 1 to Computer 14 uses only yellow cords.



Case D: (ii) and (iv) are true

In this case, the cord between Computer 13 and Computer 50 is yellow, and the cord between Computer 14 and Computer 50 is yellow, and so the route from Computer 13 to Computer 50 to Computer 14 uses only yellow cords.



Thus, if there is no route that connects Computer 1 to Computer 50 that uses only green cords, then there is always a route between Computer 13 and Computer 14 that uses only yellow cords.