The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Fermat Contest

(Grade 11)

Wednesday, February 22, 2023
(in North America and South America)

Thursday, February 23, 2023
(outside of North America and South America)

Solutions

1. Evaluating, $0.3 + 0.03 = 0.33$.

   Answer: (D)

2. Since $3 + x = 5$, then $x = 2$.
   Since $-3 + y = 5$, then $y = 8$.
   Thus, $x + y = 10$.
   Alternatively, we could have added the original two equations to obtain $(3+x)+(-3+y) = 5+5$ which simplifies to $x + y = 10$.

   Answer: (E)

3. When $x = 2$, we obtain $2x^2 + 3x^2 = 5x^2 = 5 \cdot 2^2 = 5 \cdot 4 = 20$.

   Answer: (E)

4. There are 60 minutes in an hour and 24 hours in a day.
   Thus, there are $60 \cdot 24 = 1440$ minutes in a day.
   Since there are 7 days in a week, the number of minutes in a week is $7 \cdot 1440 = 10\,080$.
   Of the given choices, this is closest to (C) 10 000.

   Answer: (C)

5. Using the given rule, the output of the machine is $2 \times 0 + 2 \times 3 = 0 + 6 = 6$.

   Answer: (D)

6. Since there are 3 doors and 2 colour choices for each door, there are $2^3 = 8$ ways of painting the three doors.
   Using "B" to represent black and "G" to represent gold, these ways are BBB, BBG, BGB, BGG, GBB, GBG, GGB, and GGG.

   Answer: (A)

7. Since juice boxes come in packs of 3, Danny needs to buy at least 6 packs for the 17 players. (If Danny bought 5 packs, he would have 15 juice boxes which is not enough; with 6 packs, he would have 18 juice boxes.)
   Since apples come in bags of 5, Danny needs to buy at least 4 bags. (We note that $3 \cdot 5 = 15$ is too small, and $4 \cdot 5 = 20$, which is enough.)
   Therefore, the minimum amount that Danny can spend is $6 \cdot \$2.00 + 4 \cdot \$4.00 = \$28.00$.

   Answer: (B)

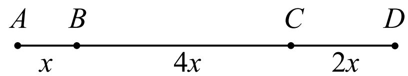
8. Riding at 15 km/h, Bri finishes the 30 km in $\dfrac{30 \text{ km}}{15 \text{ km/h}} = 2$ h.
   Riding at 20 km/h, Ari finishes the 30 km in $\dfrac{30 \text{ km}}{20 \text{ km/h}} = 1.5$ h.
   Therefore, Bri finishes 0.5 h after Ari, which is 30 minutes.

   Answer: (C)

9. In total, the three tanks contain $3600 \text{ L} + 1600 \text{ L} + 3800 \text{ L} = 9000 \text{ L}$.
   If the water is divided equally between the three tanks, each will contain $\frac{1}{3} \cdot 9000 \text{ L} = 3000 \text{ L}$.
   Therefore, $3600 \text{ L} - 3000 \text{ L} = 600 \text{ L}$ needs to be moved from Tank A to Tank B.
   (We note that 800 L would also need to be moved from Tank C to Tank B, and at this point, the three tanks will contain 3000 L.)

   Answer: (B)

10. Suppose that $AB = x$ for some $x > 0$.
Since $AB : AC = 1 : 5$, then $AC = 5x$.
This means that $BC = AC - AB = 5x - x = 4x$.
Since $BC : CD = 2 : 1$ and $BC = 4x$, then $CD = 2x$.



Therefore, $AB : CD = x : 2x = 1 : 2$.

ANSWER: (B)

11. Suppose that Mathilde had m coins at the start of last month and Salah and s coins at the start of last month.
From the given information, 100 is 25% more than m, so $100 = 1.25m$ which means that $m = \dfrac{100}{1.25} = 80$.
From the given information, 100 is 20% less than s, so $100 = 0.80s$ which means that $s = \dfrac{100}{0.80} = 125$.
Therefore, at the beginning of last month, they had a total of $m + s = 80 + 125 = 205$ coins.

ANSWER: (E)

12. A rectangle with length 8 cm and width π cm has area 8π cm^2.
Suppose that the radius of the semi-circle is r cm.
The area of a circle with radius r cm is πr^2 cm^2 and so the area of the semi-circle is $\frac{1}{2}\pi r^2$ cm^2.
Since the rectangle and the semi-circle have the same area, then $\frac{1}{2}\pi r^2 = 8\pi$ and so $\pi r^2 = 16\pi$ or $r^2 = 16$.
Since $r > 0$, then $r = 4$ and so the radius of the semi-circle is 4 cm.

ANSWER: (B)

13. The equation $a(x + 2) + b(x + 2) = 60$ has a common factor of $x + 2$ on the left side.
Thus, we can re-write the equation as $(a + b)(x + 2) = 60$.
When $a + b = 12$, we obtain $12 \cdot (x + 2) = 60$ and so $x + 2 = 5$ which gives $x = 3$.

ANSWER: (A)

14. The line with a slope of 2 and y-intercept 6 has equation $y = 2x + 6$.
To find its x-intercept, we set $y = 0$ to obtain $0 = 2x + 6$ or $2x = -6$, which gives $x = -3$.
The line with a slope of -4 and y-intercept 6 has equation $y = -4x + 6$.
To find its x-intercept, we set $y = 0$ to obtain $0 = -4x + 6$ or $4x = 6$, which gives $x = \frac{6}{4} = \frac{3}{2}$.
The distance between the points on the x-axis with coordinates $(-3, 0)$ and $(\frac{3}{2}, 0)$ is $3 + \frac{3}{2}$ which equals $\frac{6}{2} + \frac{3}{2}$ or $\frac{9}{2}$.

ANSWER: (E)

15. The 1st term is 16.
Since 16 is even, the 2nd term is $\frac{1}{2} \cdot 16 + 1 = 9$.
Since 9 is odd, the 3rd term is $\frac{1}{2}(9+1) = 5$.
Since 5 is odd, the 4th term is $\frac{1}{2}(5+1) = 3$.
Since 3 is odd, the 5th term is $\frac{1}{2}(3+1) = 2$.
Since 2 is even, the 6th term is $\frac{1}{2} \cdot 2 + 1 = 2$.
This previous step shows us that when one term is 2, the next term will also be 2.
Thus, the remaining terms in this sequence are all 2.
In particular, the 101st term is 2.

ANSWER: (B)

16. The given arrangement has 14 zeroes and 11 ones showing.
Loron can pick any row or column in which to flip the 5 cards over. Furthermore, the row or column that Loron chooses can contain between 0 and 5 of the cards with different numbers on their two sides.
Of the 5 rows and 5 columns, 3 have 4 zeroes and 1 one, 2 have 3 zeroes and 2 ones, and 5 have 2 zeroes and 3 ones.
This means that the number of zeroes cannot decrease by more than 4 when the cards in a row or column are flipped, since the only way that the zeroes could decrease by 5 is if all five cards in the row or column had 0 on the top face and 1 on the bottom face.
Therefore, there cannot be as few as $14 - 5 = 9$ zeroes after Loron flips the cards, which means that the ratio cannot be $9 : 16$, or (C). This means that the answer to the given problem is (C).

For completeness, we will show that the other ratios are indeed achievable.
If Loron chooses the first column and if this column includes 3 cards with ones on both sides, and 2 cards with zeroes on one side (facing up) and ones on the reverse side, then flipping the cards in this column yields $14 - 2 = 12$ zeroes and $11 + 2 = 13$ ones.
Thus, the ratio $12 : 13$ (choice (A)) is possible.
If Loron chooses the fifth column and if this column includes 1 card with a one on both sides and 4 cards with zeroes on one side (facing up) and ones on the reverse side, then flipping the cards in this column yields $14 - 4 = 10$ zeroes and $11 + 4 = 15$ ones.
Thus, the ratio $10 : 15 = 2 : 3$ (choice (B)) is possible.
If Loron chooses the first column and if the top 4 cards in this column have the same numbers on both sides and the bottom card has a one on the top side and a zero on the reverse side, then flipping the cards in this column yields $14 + 1 = 15$ zeroes and $11 - 1 = 10$ ones.
Thus, the ratio $15 : 10 = 3 : 2$ (choice (D)) is possible.
If Loron chooses the first column and if the first, fourth and fifth cards in this column have the same numbers on both sides and the second and third cards each has a one on the top side and a zero on the reverse side, then flipping the cards in this column yields $14 + 2 = 16$ zeroes and $11 - 2 = 9$ ones.
Thus, the ratio $16 : 9$ (choice (E)) is possible.
Therefore, the only ratio of the five that are given that is not possible is $9 : 16$, or (C).

ANSWER: (C)

17. We start by finding the prime factors of 1184:

$$1184 = 2 \cdot 592 = 2^2 \cdot 296 = 2^3 \cdot 148 = 2^4 \cdot 74 = 2^5 \cdot 37$$

The positive divisors of 1184 cannot contain prime factors other than 2 and 37, and cannot contain more than 5 factors of 2 or 1 factor of 37.
Thus, the positive divisors are

$$1, 2, 4, 8, 16, 32, 37, 74, 148, 296, 592, 1184$$

(The first five of these divisors have 0 factors of 37 and 0 through 5 factors of 2, while the last five have 1 factor 37 and 0 through 5 factors of 2.)
The sum, S, of these divisors is

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + 16 + 32 + 37 + 74 + 148 + 296 + 592 + 1184 \\ &= (1 + 2 + 4 + 8 + 16 + 32) + 37 \cdot (1 + 2 + 4 + 8 + 16 + 32) \\ &= (1 + 2 + 4 + 8 + 16 + 32) \cdot (1 + 37) \\ &= 63 \cdot 38 \\ &= 2394 \end{aligned}$$

ANSWER: (A)

18. Each group of four jumps takes the grasshopper 1 cm to the east and 3 cm to the west, which is a net movement of 2 cm to the west, and 2 cm to the north and 4 cm to the south, which is a net movement of 2 cm to the south.
In other words, we can consider each group of four jumps, starting with the first, as resulting in a net movement of 2 cm to the west and 2 cm to the south.
We note that $158 = 2 \times 79$.
Thus, after 79 groups of four jumps, the grasshopper is $79 \times 2 = 158$ cm to the west and 158 cm to the south of its original position. (We need at least 79 groups of these because the grasshopper cannot be 158 cm to the south of its original position before the end of 79 such groups.)
The grasshopper has made $4 \times 79 = 316$ jumps so far.
After the 317th jump (1 cm to the east), the grasshopper is 157 cm west and 158 cm south of its original position.
After the 318th jump (2 cm to the north), the grasshopper is 157 cm west and 156 cm south of its original position.
After the 319th jump (3 cm to the west), the grasshopper is 160 cm west and 156 cm south of its original position.
After the 320th jump (4 cm to the south), the grasshopper is 160 cm west and 160 cm south of its original position.
After the 321st jump (1 cm to the east), the grasshopper is 159 cm west and 160 cm south of its original position.
After the 322nd jump (2 cm to the north), the grasshopper is 159 cm west and 158 cm south of its original position.
After the 323rd jump (3 cm to the west), the grasshopper is 162 cm west and 158 cm south of its original position, which is the desired position.
As the grasshopper continues jumping, each of its positions will always be at least 160 cm south of its original position, so this is the only time that it is at this position.
Therefore, $n = 323$. The sum of the squares of the digits of n is $3^2 + 2^2 + 3^2 = 9 + 4 + 9 = 22$.

ANSWER: (A)

19. If x and y satisfy $2x^2 + 8y = 26$, then $x^2 + 4y = 13$ and so $4y = 13 - x^2$.
Since x and y are integers, then $4y$ is even and so $13 - x^2$ is even, which means that x is odd.
Since x is odd, we can write $x = 2q + 1$ for some integer q.
Thus, $4y = 13 - x^2 = 13 - (2q + 1)^2 = 13 - (4q^2 + 4q + 1) = 12 - 4q^2 - 4q$.
Since $4y = 12 - 4q^2 - 4q$, then $y = 3 - q^2 - q$.
Thus, $x - y = (2q + 1) - (3 - q^2 - q) = q^2 + 3q - 2$.
When $q = 4$, we obtain $x - y = q^2 + 3q - 2 = 4^2 + 3 \cdot 4 - 2 = 26$.
We note also that, when $q = 4$, $x = 2q + 1 = 9$ and $y = 3 - q^2 - q = -17$ which satisfy $x^2 + 4y = 13$.
We can also check that there is no integer q for which $q^2 + 3q - 2$ is equal to any of -8, -16, 22, or 30. (For example, if $q^2 + 3q - 2 = -16$, then $q^2 + 3q + 14 = 0$, and this quadratic equation has no integer solutions.)

ANSWER: (B)

20. If $n!$ ends with exactly m zeroes, then $n!$ is divisibe by 10^m but not divisible by 10^{m+1}. (If $n!$ were divisible by 10^{m+1}, it would end with at least $m + 1$ zeroes.)
In this case, we can write $n! = 10^m \cdot q$ where q is not divisible by 10. This in turn means that either q is not divisible by 2 or not divisible by 5 or both.
Since $2 < 5$, when $n \geq 2$, the product $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ includes more multiples of 2 than of 5 among the n integers in its product, so $n!$ includes more factors of 2 than of 5.
This in turn means that, if $n!$ ends in exactly m zeroes, then $n! = 10^m \cdot q$ where q is not divisible by 5, and so the number of zeroes at the end of $n!$ is exactly equal to the number of prime factors of 5 in the prime factorization of $n!$.
We note also that as n increases, the number of zeroes at the end of $n!$ never decreases since the number of factors of 5 either stays the same or increases as n increases.
For $n = 1$ to $n = 4$, the product $n!$ includes 0 multiples of 5, so $n!$ ends in 0 zeroes.
For $n = 5$ to $n = 9$, the product $n!$ includes 1 multiple of 5 (namely 5), so $n!$ ends in 1 zero.
For $n = 10$ to $n = 14$, the product $n!$ includes 2 multiples of 5 (namely 5 and 10), so $n!$ ends in 2 zeroes.
For $n = 15$ to $n = 19$, the product $n!$ includes 3 multiples of 5 (namely 5, 10 and 15), so $n!$ ends in 3 zeroes.
For $n = 20$ to $n = 24$, the product $n!$ includes 4 multiples of 5 (namely 5, 10, 15, and 20), so $n!$ ends in 4 zeroes.
For $n = 25$ to $n = 29$, the product $n!$ includes 5 multiples of 5 (namely 5, 10, 15, 20, and 25) and includes 6 factors of 5 (since 25 contributes 2 factors of 5), so $n!$ ends in 6 zeroes.
For $n = 30$ to $n = 34$, $n!$ ends in 7 zeroes. For $n = 35$ to $n = 39$, $n!$ ends in 8 zeroes.
For $n = 40$ to $n = 44$, $n!$ ends in 9 zeroes. For $n = 45$ to $n = 49$, $n!$ ends in 10 zeroes.
For $n = 50$ to $n = 54$, $n!$ ends in 12 zeroes, since the product $n!$ includes 10 multiples of 5, two of which include 2 factors of 5.
For $n = 55$ to $n = 74$, $n!$ will end in 13, 14, 15, 16 zeroes as n increases.
For $n = 75$ to $n = 79$, $n!$ ends in 18 zeroes.
For $n = 80$ to $n = 99$, $n!$ ends of 19, 20, 21, 22 zeroes as n increases.
For $n = 100$ to $n = 104$, $n!$ ends in 24 zeroes.
For $n = 105$ to $n = 124$, $n!$ ends in 25, 26, 27, 28 zeroes.
For $n = 125$, $n!$ ends in 31 zeroes since 125 includes 3 factors of 5, so 125! ends in 3 more than zeroes than 124!.
Of the integers m with $1 \leq m \leq 30$, there is no value of n for which $n!$ ends in m zeroes when $m = 5, 11, 17, 23, 29, 30$, which means that $30 - 6 = 24$ of the values of m are possible.

ANSWER: (D)

21. From the given information, if a and b are in two consecutive squares, then $a+b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a+b$ is larger than both a and b.
This means that the largest integer in the list, which is 13, cannot be either x or y (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (because 13 is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x+y$ to be as large as possible, we would have x and y equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x+y$ is when $x=9$ and $y=11$. (We could also swap x and y.)
Here, we could have $13=11+2$ and $10=9+1$, giving the following partial list:

$$\boxed{11}\quad 13\quad \boxed{2}\quad \bigcirc\quad \boxed{}\quad \bigcirc\quad \boxed{1}\quad 10\quad \boxed{9}$$

The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.

$$\boxed{11}\quad 13\quad \boxed{2}\quad 6\quad \boxed{4}\quad 5\quad \boxed{1}\quad 10\quad \boxed{9}$$

This tells us that the largest possible value of $x+y$ is 20.

ANSWER: 20

22. *Solution 1*
Starting with the given relationship between x and y and manipulating algebraically, we obtain successively

$$\begin{aligned} \frac{1}{x+y} &= \frac{1}{x}-\frac{1}{y} \\ xy &= (x+y)y-(x+y)x \qquad \text{(multiplying by } xy(x+y)\text{)} \\ xy &= xy+y^2-x^2-xy \\ x^2+xy-y^2 &= 0 \\ \frac{x^2}{y^2}+\frac{x}{y}-1 &= 0 \qquad \text{(dividing by } y^2 \text{ which is non-zero)} \\ t^2+t-1 &= 0 \end{aligned}$$

where $t=\dfrac{x}{y}$.
Since $x>0$ and $y>0$, then $t>0$. Using the quadratic formula

$$t=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}=\frac{-1\pm\sqrt{5}}{2}$$

Since $t > 0$, then $\dfrac{x}{y} = t = \dfrac{\sqrt{5}-1}{2}$.

Therefore,

$$\begin{aligned}
\left(\frac{x}{y} + \frac{y}{x}\right)^2 &= \left(\frac{\sqrt{5}-1}{2} + \frac{2}{\sqrt{5}-1}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{2(\sqrt{5}+1)}{4}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{\sqrt{5}+1}{2}\right)^2 \\
&= (\sqrt{5})^2 \\
&= 5
\end{aligned}$$

Solution 2

Since $x, y > 0$, the following equations are equivalent:

$$\begin{aligned}
\frac{1}{x+y} &= \frac{1}{x} - \frac{1}{y} \\
1 &= \frac{x+y}{x} - \frac{x+y}{y} \\
1 &= \frac{x}{x} + \frac{y}{x} - \frac{x}{y} - \frac{y}{y} \\
1 &= 1 + \frac{y}{x} - \frac{x}{y} - 1 \\
1 &= \frac{y}{x} - \frac{x}{y} \\
-1 &= \frac{x}{y} - \frac{y}{x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\frac{x}{y} + \frac{y}{x}\right)^2 &= \frac{x^2}{y^2} + 2 \cdot \frac{x}{y} \cdot \frac{y}{x} + \frac{y^2}{x^2} \\
&= \frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} \\
&= \frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} + 4 \\
&= \frac{x^2}{y^2} - 2 \cdot \frac{x}{y} \cdot \frac{y}{x} + \frac{y^2}{x^2} + 4 \\
&= \left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4 \\
&= (-1)^2 + 4 \\
&= 5
\end{aligned}$$

ANSWER: 05

23. We write an integer n with $100 \leq n \leq 999$ as $n = 100a + 10b + c$ for some digits a, b and c.
That is, n has hundreds digit a, tens digit b, and ones digit c.
For each such integer n, we have $s(n) = a + b + c$.
We want to count the number of such integers n with $7 \leq a + b + c \leq 11$.
When $100 \leq n \leq 999$, we know that $1 \leq a \leq 9$ and $0 \leq b \leq 9$ and $0 \leq c \leq 9$.

First, we count the number of n with $a + b + c = 7$.
If $a = 1$, then $b + c = 6$ and there are 7 possible pairs of values for b and c. These pairs are $(b, c) = (0, 6), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)$.
If $a = 2$, then $b + c = 5$ and there are 6 possible pairs of values for b and c.
Similarly, when $a = 3, 4, 5, 6, 7$, there are 5, 4, 3, 2, 1 pairs of values, respectively, for b and c.
In other words, the number of integers n with $a+b+c = 7$ is equal to $7+6+5+4+3+2+1 = 28$.

Using a similar process, we can determine that the number of such integers n with $s(n) = 8$ is $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ and the number of such integers n with $s(n) = 9$ is $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$.

We have to be more careful counting the number of integers n with $s(n) = 10$ and $s(n) = 11$, because none of the digits can be greater than 9.
Consider the integers n with $a + b + c = 10$.
If $a = 1$, then $b + c = 9$ and there are 10 possible pairs of values for b and c. These pairs are $(b, c) = (0, 9), (1, 8), \ldots, (8, 1), (9, 0)$.
If $a = 2$, then $b + c = 8$ and there are 9 possible pairs of values for b and c.
As a increases from 1 to 9, we find that there are $10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 54$ such integers n.
(Note that when $a = 9$, we have $b + c = 1$ and there are 2 pairs of values for b and c.)
Finally, we consider the integers n with $a + b + c = 11$.
If $a = 1$, then $b + c = 10$ and there are 9 possible pairs of values for b and c. These pairs are $(b, c) = (1, 9), (2, 8), \ldots, (8, 2), (9, 1)$.
If $a = 2$, then $b + c = 9$ and there are 10 possible pairs of values for b and c.
If $a = 3$, then $b + c = 8$ and there are 9 possible pairs of values for b and c.
Continuing in this way, we find that there are $9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 = 61$ such integers n.

Having considered all cases, we see that the number of such integers n is

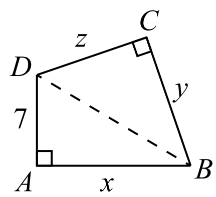
$$S = 28 + 36 + 45 + 54 + 61 = 224$$

The rightmost two digits of S are 24.

ANSWER: 24

24. *Solution 1*

Suppose that $AB = x$, $BC = y$, $CD = z$, and $DA = 7$. (It does not matter to which side length we assign the fixed length of 7.)
We are told that x, y and z are integers.
Since the perimeter of $ABCD$ is 224, we have $x + y + z + 7 = 224$ or $x + y + z = 217$.
Join B to D.



The area of $ABCD$ is equal to the sum of the areas of $\triangle DAB$ and $\triangle BCD$.
Since these triangles are right-angled, then $2205 = \frac{1}{2} \cdot DA \cdot AB + \frac{1}{2} \cdot BC \cdot CD$.
Multiplying by 2, we obtain $4410 = 7x + yz$.
Finally, we also note that, using the Pythagorean Theorem twice, we obtain

$$DA^2 + AB^2 = DB^2 = BC^2 + CD^2$$

and so $49 + x^2 = y^2 + z^2$.
We need to determine the value of $S = x^2 + y^2 + z^2 + 7^2$.
Since $x + y + z = 217$, then $x = 217 - y - z$.
Substituting into $4410 = 7x + yz$ and proceeding algebraically, we obtain successively

$$\begin{aligned}
4410 &= 7x + yz \\
4410 &= 7(217 - y - z) + yz \\
4410 &= 1519 - 7y - 7z + yz \\
2891 &= yz - 7y - 7z \\
2891 &= y(z - 7) - 7z \\
2891 &= y(z - 7) - 7z + 49 - 49 \\
2940 &= y(z - 7) - 7(z - 7) \\
2940 &= (y - 7)(z - 7)
\end{aligned}$$

Therefore, $y - 7$ and $z - 7$ form a positive divisor pair of 2940. (Since their product is postiive, they are either both positive or both negative. Since y and z are positive, if both of $y - 7$ and $z - 7$ are negative, we would have $0 < y < 7$ and $0 < z < 7$ which could not be large enough to allow for a feasible value of x.)
We note that $y + z = 217 - x$ and so $y + z < 217$ which means that $(y - 7) + (z - 7) < 203$.
Since

$$2940 = 20 \cdot 147 = 2^2 \cdot 5 \cdot 3 \cdot 7^2$$

then the divisors of 2940 are the positive integers of the form $2^r \cdot 3^s \cdot 5^t \cdot 7^u$ where $0 \leq r \leq 2$ and $0 \leq s \leq 1$ and $0 \leq t \leq 1$ and $0 \leq u \leq 2$.
Thus, these divisors are

$$1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 49,$$

$$60, 70, 84, 98, 105, 140, 147, 196, 210, 245, 294, 420, 490, 588, 735, 980, 1470, 2940$$

We can remove divisor pairs from this list whose sum is greater than 203. This gets us to the shorter list

$$20, 21, 28, 30, 35, 42, 49, 60, 70, 84, 98, 105, 140, 147$$

This means that there are 7 divisor pairs remaining to consider. We can assume that $y < z$. Using the fact that $x + y + z = 217$, we can solve for x in each case. These values of x, y and z will satisfy the perimeter and area conditions, but we need to check the Pythaogrean condition. We make a table:

$y-7$	$z-7$	y	z	$x = 217 - y - z$	$y^2 + z^2 - x^2$
20	147	27	154	36	23 149
21	140	28	147	42	20 629
28	105	35	112	70	8869
30	98	37	105	75	6769
35	84	42	91	84	2989
42	70	49	77	91	49
49	60	56	67	94	−1211

Since we need $y^2 + z^2 - x^2 = 49$, then we must have $y = 49$ and $z = 77$ and $x = 91$.
This means that $S = x^2 + y^2 + z^2 + 7^2 = 91^2 + 49^2 + 77^2 + 7^2 = 16\,660$.
The rightmost two digits of S are 60.

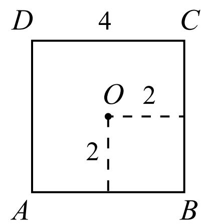
Solution 2
As in Solution 1, we have $x + y + z = 217$ and $4410 = 7x + yz$ and $x^2 + 49 = y^2 + z^2$.
Re-arranging and squaring the first equation and using the second and third equations, we obtain

$$\begin{aligned}
y + z &= 217 - x \\
y^2 + z^2 + 2yz &= x^2 - 434x + 217^2 \\
(x^2 + 49) + 2(4410 - 7x) &= x^2 - 434x + 217^2 \\
49 + 8820 - 14x &= -434x + 217^2 \\
420x &= 217^2 - 8820 - 49 \\
420x &= 38\,220 \\
x &= 91
\end{aligned}$$

Thus, $y + z = 217 - 91 = 126$ and $yz = 4410 - 7 \cdot 91 = 3773$.
This gives, $y(126 - y) = 3773$ and so $y^2 - 126y + 3773 = 0$ or $(y - 49)(y - 77) = 0$.
Therefore, $y = 49$ (which means $z = 77$) or $y = 77$ (which means $z = 49$).
We note that $y^2 + z^2 = 49^2 + 77^2 = 8330 = 91^2 + 7^2 = x^2 + 7^2$ which verifies the remaining equation.
This means that $S = x^2 + y^2 + z^2 + 7^2 = 91^2 + 49^2 + 77^2 + 7^2 = 16\,660$.
The rightmost two digits of S are 60.

ANSWER: 60

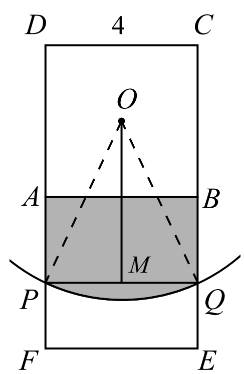
25. Throughout this solution, we will not explicitly include units, but will assume that all lengths are in metres and all areas are in square metres.
The top face of the cube is a square, which we label $ABCD$, and we call its centre O.
Since the cube has edge length 4, then the side length of square $ABCD$ is 4.
This means that O is a perpendicular distance of 2 from each of the sides of square $ABCD$, and thus is a distance of $\sqrt{2^2+2^2}=\sqrt{8}$ from each of the vertices of $ABCD$.



These vertices are the farthest points on $ABCD$ from O.
Since $\sqrt{8}\approx 2.8$, then the loose end of the rope of length 5 can reach every point on $ABCD$, which has area 16.

Next, the rope cannot reach to the bottom face of the cube because the shortest distance along the surface of the cube from O to the bottom face is 6 and the rope has length 5. We will confirm this in another way shortly.
Also, since the rope is anchored to the centre of the top face and all of the faces are square, the rope can reach the same area on each of the four side faces.
Suppose that the area of one of the side faces that can be reached is a. Since the rope can reach the entire area of the top face, then the total area that can be reached is $16+4a$.
We thus need to determine the value of a.

Suppose that one of the side faces is square $ABEF$, which has side length 4. Consider the figure created by square $ABCD$ and square $ABEF$ together. We can think of this as an "unfolding" of part of the cube.



When the rope is stretched tight, its loose end traces across square $ABEF$ an arc of a circle centred at O and with radius 5.
Notice that the farthest that the rope can reach down square $ABEF$ is a distance of 3, since its anchor is a distance of 2 from AB. This confirms that the rope cannot reach the bottom face of the cube since it would have to cross FE to do so.
Suppose that this arc cuts AF at P and cuts BE at Q.

We want to determine the area of square $ABEF$ above arc PQ (the shaded area); the area of this region is a.

We will calculate the value of a by determining the area of rectangle $ABQP$ and adding the area of the region between the circular arc and line segment PQ.
We will calculate this latter area by determining the area of sector OPQ and subtracting the area of $\triangle OPQ$.
We note that $PQ = 4$. Let M be the midpoint of PQ; thus $PM = MQ = 2$.
Since $\triangle OPQ$ is isosceles with $OP = OQ = 5$, then OM is perpendicular to PQ.
By the Pythagorean Theorem, $OM = \sqrt{OP^2 - PM^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$.
Thus, the area of $\triangle OPQ$ is $\frac{1}{2} \cdot PQ \cdot OM = \frac{1}{2} \cdot 4 \cdot \sqrt{21} = 2\sqrt{21}$.
Furthermore, since O is a distance of 2 from AB and $OM = \sqrt{21}$, then the height of rectangle $ABQP$ is $\sqrt{21} - 2$.
Thus, the area of rectangle $ABQP$ is $4 \cdot (\sqrt{21} - 2) = 4\sqrt{21} - 8$.
To find the area of sector OPQ, we note that the area of a circle with radius 5 is $\pi \cdot 5^2$, and so the area of the sector is $\dfrac{\angle POQ}{360^\circ} \cdot 25\pi$.
Now, $\angle POQ = 2\angle POM = 2\sin^{-1}(2/5)$, since $\triangle POM$ is right-angled at M which means that $\sin(\angle POM) = \dfrac{PM}{OP}$.
Thus, the area of the sector is $\dfrac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi$.
Putting this all together, we obtain

$$\begin{aligned}
100A &= 100(16 + 4a) \\
&= 1600 + 400a \\
&= 1600 + 400\left((4\sqrt{21} - 8) + \frac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi - 2\sqrt{21}\right) \\
&= 1600 + 400\left(2\sqrt{21} - 8 + \frac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi\right) \\
&= 800\sqrt{21} - 1600 + \frac{800\sin^{-1}(2/5) \cdot 25\pi}{360^\circ} \\
&\approx 6181.229
\end{aligned}$$

(Note that we have not switched to decimal approximations until the very last step in order to avoid any possible rounding error.)
Therefore, the integer closest to $100A$ is 6181, whose rightmost two digits are 81.

Answer: 81