The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 4, 2023
(in North America and South America)

Wednesday, April 5, 2023
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) The average of n, $2n$, $3n$, $4n$, $5n$ is 18. What is the value of n?

   (b) Suppose that $2x + y = 5$ and $x + 2y = 7$. What is the average of x and y?

   (c) The average of t^2, $2t$ and 3 is 9. If $t < 0$, determine the value of t.

2. (a) If $Q(5, 3)$ is the midpoint of the line segment with endpoints $P(1, p)$ and $R(r, 5)$, what are the values of p and r?

   (b) A line with slope 3 and another line with slope -1 intersect at $P(3, 6)$. What is the distance between the x-intercepts of the two lines?

   (c) For some value of t, the line with equation $y = tx + t$ is perpendicular to the line with equation $y = 2x + 7$. Determine the point of intersection of these two lines.

3. (a) The positive divisors of 6 are 1, 2, 3, and 6. What is the sum of the positive divisors of 64?

   (b) Fionn wrote 4 consecutive integers on a whiteboard. Lexi came along and erased one of the integers. Fionn noticed that the sum of the remaining integers was 847. What integer did Lexi erase?

   (c) An arithmetic sequence with 7 terms has first term d^2 and common difference d. The sum of the 7 terms in the sequence is 756. Determine all possible values of d.

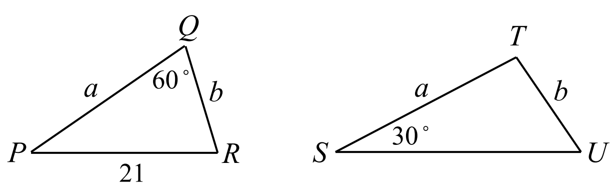
   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

4. (a) Liang and Edmundo paint at different but constant rates. Liang can paint a room in 3 hours if she works alone. Edmundo can paint the same room in 4 hours if he works alone. Liang works alone for 2 hours and then stops. Edmundo finishes painting the room. How many minutes will Edmundo need to finish painting the room?

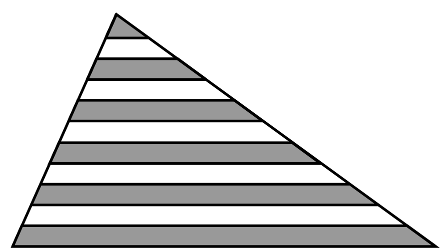
(b) On January 1, 2021, an investment had a value of \$400.
From January 1, 2021 to January 1, 2022, the value of the investment increased by $A\%$ from its value on January 1, 2021 for some $A > 0$.
From January 1, 2022 to January 1, 2023, the value of the investment decreased by $A\%$ from its value on January 1, 2022.
On January 1, 2023, the value of the investment was \$391.
Determine all possible values of A.

5. (a) Suppose that $f(x) = x^2 + (2n-1)x + (n^2 - 22)$ for some integer n. What is the smallest positive integer n for which $f(x)$ has no real roots?

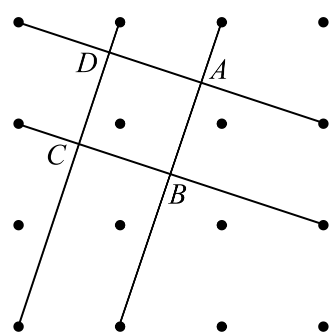
(b) In the diagram, $\triangle PQR$ has $PQ = a$, $QR = b$, $PR = 21$, and $\angle PQR = 60^\circ$. Also, $\triangle STU$ has $ST = a$, $TU = b$, $\angle TSU = 30^\circ$, and $\sin(\angle TUS) = \frac{4}{5}$. Determine the values of a and b.



6. (a) A triangle of area 770 cm^2 is divided into 11 regions of equal height by 10 lines that are all parallel to the base of the triangle. Starting from the top of the triangle, every other region is shaded, as shown. What is the total area of the shaded regions?



(b) A square lattice of 16 points is constructed such that the horizontal and vertical distances between adjacent points are all exactly 1 unit. Each of four pairs of points are connected by a line segment, as shown. The intersections of these line segments are the vertices of square $ABCD$. Determine the area of square $ABCD$.



7. (a) A bag contains 3 red marbles and 6 blue marbles. Akshan removes one marble at a time until the bag is empty. Each marble that they remove is chosen randomly from the remaining marbles. Given that the first marble that Akshan removes is red and the third marble that they remove is blue, what is the probability that the last two marbles that Akshan removes are both blue?

(b) Determine the number of quadruples of positive integers (a, b, c, d) with $a < b < c < d$ that satisfy both of the following system of equations:

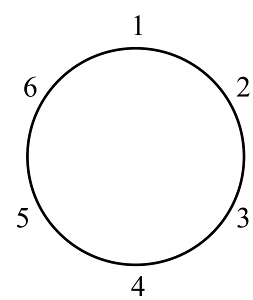
$$\begin{aligned} ac + ad + bc + bd &= 2023 \\ a + b + c + d &= 296 \end{aligned}$$

8. (a) Suppose that $\triangle ABC$ is right-angled at B and has $AB = n(n+1)$ and $AC = (n+1)(n+4)$, where n is a positive integer. Determine the number of positive integers $n < 100\,000$ for which the length of side BC is also an integer.

(b) Determine all real values of x for which

$$\sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = 4$$

9. At the Canadian Eatery with Multiple Configurations, there are round tables, around which chairs are placed. When a table has n chairs around it for some integer $n \geq 3$, the chairs are labelled $1, 2, 3, \ldots, n-1, n$ in order around the table. A table is considered full if no more people can be seated without having two people sit in neighbouring chairs. For example, when $n = 6$, full tables occur when people are seated in chairs labelled $\{1, 4\}$ or $\{2, 5\}$ or $\{3, 6\}$ or $\{1, 3, 5\}$ or $\{2, 4, 6\}$. Thus, there are 5 different full tables when $n = 6$.



(a) Determine all ways in which people can be seated around a table with 8 chairs so that the table is full, in each case giving the labels on the chairs in which people are sitting.

(b) A full table with $6k + 5$ chairs, for some positive integer k, has t people seated in its chairs. Determine, in terms of k, the number of possible values of t.

(c) Determine the number of different full tables when $n = 19$.

10. For every real number x, define $\lfloor x \rfloor$ to be equal to the greatest integer less than or equal to x. (We call this the "floor" of x.) For example, $\lfloor 4.2 \rfloor = 4$, $\lfloor 5.7 \rfloor = 5$, $\lfloor -3.4 \rfloor = -4$, $\lfloor 0.4 \rfloor = 0$, and $\lfloor 2 \rfloor = 2$.

(a) Determine the integer equal to $\left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{59}{3} \right\rfloor + \left\lfloor \frac{60}{3} \right\rfloor$.

(The sum has 60 terms.)

(b) Determine a polynomial $p(x)$ so that for every positive integer $m > 4$,

$$\lfloor p(m) \rfloor = \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{m-1}{3} \right\rfloor$$

(The sum has $m - 1$ terms.)

A *polynomial* $f(x)$ is an algebraic expression of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some integer $n \geq 0$ and for some real numbers $a_n, a_{n-1}, \ldots, a_1, a_0$.

(c) For each integer $n \geq 1$, define $f(n)$ to be equal to an infinite sum:

$$f(n) = \left\lfloor \frac{n}{1^2+1} \right\rfloor + \left\lfloor \frac{2n}{2^2+1} \right\rfloor + \left\lfloor \frac{3n}{3^2+1} \right\rfloor + \left\lfloor \frac{4n}{4^2+1} \right\rfloor + \left\lfloor \frac{5n}{5^2+1} \right\rfloor + \cdots$$

(The sum contains the terms $\left\lfloor \frac{kn}{k^2+1} \right\rfloor$ for all positive integers k, and no other terms.)

Suppose $f(t+1) - f(t) = 2$ for some odd positive integer t. Prove that t is a prime number.

2023 Euclid Contest (English)

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For students...

Thank you for writing the 2023 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2023 Canadian Senior Mathematics Contest, which will be written in November 2023.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

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- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results