The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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# *2023 Cayley Contest*

**(Grade 10)**

**Wednesday, February 22, 2023**
**(in North America and South America)**

**Thursday, February 23, 2023**
**(outside of North America and South America)**

*Solutions*

1. Since each of the three fractions is equal to 1, then $\frac{1}{1} + \frac{2}{2} + \frac{3}{3} = 1 + 1 + 1 = 3$.

   Answer: (C)

2. Since $3n = 9 + 9 + 9 = 3 \times 9$, then $n = 9$.
   Alternatively, we could note that $9 + 9 + 9 = 27$ and so $3n = 27$ which gives $n = \frac{27}{3} = 9$.

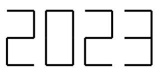
   Answer: (D)

3. We add 25 minutes to 1 hour and 48 minutes in two steps.
   First, we add 12 minutes to 1 hour and 48 minutes to get 2 hours.
   Then we add $25 - 12 = 13$ minutes to 2 hours to get 2 hours and 13 minutes.
   Alternatively, we could note that 1 hour and 48 minutes is $60 + 48 = 108$ minutes, and so the time that is 25 minutes longer is 133 minutes, which is $120 + 13$ minutes or 2 hours and 13 minutes.

   Answer: (A)

4. On Day 1, Lucy sees 2 blue jays and 3 cardinals, and so sees 1 more cardinal than blue jay.
   On Day 2, Lucy sees 3 blue jays and 3 cardinals.
   On Day 3, Lucy sees 2 blue jays and 4 cardinals, and so sees 2 more cardinals than blue jays.
   Thus, over the three days, Lucy saw $1 + 0 + 2 = 3$ more cardinals than blue jays.

   Answer: (B)

5. When looking at  through a window from behind, the digits appear as mirror images and in reverse order, and so appear as .

   Answer: (C)

6. *Solution 1*
   Since $\angle BCD$ is a straight angle, then $\angle ACB = 180^\circ - \angle ACD = 180^\circ - 150^\circ = 30^\circ$.
   Since the measures of the angles in $\triangle ABC$ add to $180^\circ$, then $x^\circ + 90^\circ + 30^\circ = 180^\circ$ and so $x = 180 - 120 = 60$.

   *Solution 2*
   Since the measures of the angles in $\triangle ABC$ add to $180^\circ$, then

   $$\angle ACB = 180^\circ - \angle ABC - \angle BAC = 180^\circ - 90^\circ - x^\circ = 90^\circ - x^\circ$$

   Since $\angle BCD = 180^\circ$, then

   $$\begin{aligned} \angle ACB + \angle ACD &= 180^\circ \\ (90^\circ - x^\circ) + 150^\circ &= 180^\circ \\ 240 - x &= 180 \end{aligned}$$

   and so $x = 60$.

   Answer: (E)

7. A cube has six identical faces.
   If the surface area of a cube is 24, the area of each face is $\frac{24}{6} = 4$.
   Since each face of this cube is a square with area 4, the edge length of the cube is $\sqrt{4} = 2$.
   Thus, the volume of the cube is $2^3$ which equals 8.

   Answer: (E)

8. If $\frac{4}{5}$ of the beads are yellow, then $\frac{1}{5}$ are green.
Since there are 4 green beads, the total number of beads must be $4 \times 5 = 20$.
Thus, Charlie needs to add $20 - 4 = 16$ yellow beads.

Answer: (A)

9. *Solution 1*
Suppose that the original number is 100.
When 100 is increased by 60%, the result is 160.
To return to the original value of 100, 160 must be decreased by 60.
This percentage is $\frac{60}{160} \times 100\% = \frac{3}{8} \times 100\% = 37.5\%$.

*Solution 2*
Suppose that the original number is $x$ for some $x > 0$.
When $x$ is increased by 60%, the result is $1.6x$.
To return to the original value of $x$, $1.6x$ must be decreased by $0.6x$.
This percentage is $\frac{0.6x}{1.6x} \times 100\% = \frac{3}{8} \times 100\% = 37.5\%$.

Answer: (E)

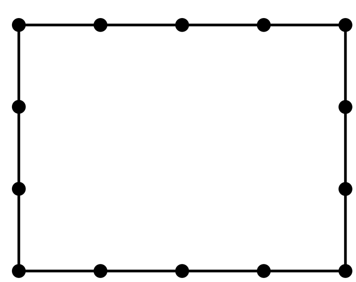
10. Since each door can be open or closed, there are 2 possible states for each door.
Since there are 5 doors, there are $2^5 = 32$ combinations of states for the 5 doors.
If the doors are labelled P, Q, R, S, T, the pairs of doors that can be opened are PQ, PR, PS, PT, QR, QS, QT, RS, RT, ST. There are 10 such pairs.
Therefore, if one of the 32 combinations of states is chosen at random, the probability that exactly two doors are open is $\frac{10}{32}$ which is equivalent to $\frac{5}{16}$.

Answer: (A)

11. After Karim eats $n$ candies, he has $23 - n$ candies remaining.
Since he divides these candies equally among his three children, the integer $23 - n$ must be a multiple of 3.
If $n = 2, 5, 11, 14$, we obtain $23 - n = 21, 18, 12, 9$, each of which is a multiple of 3.
If $n = 9$, we obtain $23 - n = 14$, which is not a multiple of 3.
Therefore, $n$ cannot equal 9.

Answer: (C)

12. A rectangle that is 6 m by 8 m has perimeter $2 \times (6 \text{ m} + 8 \text{ m}) = 28$ m.
If posts are put in every 2 m around the perimeter starting at a corner, then we would guess that it will take $\dfrac{28 \text{ m}}{2 \text{ m}} = 14$ posts.
The diagram below confirms this:

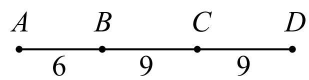


Answer: (B)

13. Since $2023 = 7 \times 17^2$, then any perfect square that is a multiple of 2023 must have prime factors of both 7 and 17.
Furthermore, the exponents of the prime factors of a perfect square must be all even.
Therefore, any perfect square that is a multiple of 2023 must be divisible by $7^2$ and by $17^2$, and so it is at least $7^2 \times 17^2$ which equals $7 \times 2023$.
Therefore, the smallest perfect square that is a multiple of 2023 is $7 \times 2023$.
We can check that $2023^2$ is larger than $7 \times 2023$ and that none of $4 \times 2023$ and $17 \times 2023$ and $7 \times 17 \times 2023$ is a perfect square.

ANSWER: (C)

14. Since $B$ is between $A$ and $D$ and $BD = 3AB$, then $B$ splits $AD$ in the ratio $1 : 3$.
Since $AD = 24$, then $AB = 6$ and $BD = 18$.
Since $C$ is halfway between $B$ and $D$, then $BC = \frac{1}{2}BD = 9$.




Thus, $AC = AB + BC = 6 + 9 = 15$.

ANSWER: (E)

15. Since $a = \dfrac{1}{n}$ where $n$ is a positive integer with $n > 1$, then $0 < a < 1$ and $\dfrac{1}{a} = n > 1$.
Thus, $0 < a < 1 < \dfrac{1}{a}$, which eliminates choices (D) and (E).
Since $0 < a < 1$, then $a^2$ is positive and $a^2 < a$, which eliminates choices (A) and (C).
Thus, $0 < a^2 < a < 1 < \dfrac{1}{a}$, which tells us that (B) must be correct.

ANSWER: (B)

16. *Solution 1*
Since $AB$ and $ED$ are parallel, quadrilateral $ABDE$ is a trapezoid.
We know that $AB = 30$ cm.
Since $ABCF$ is a rectangle, then $FC = AB = 30$ cm.
Suppose that $DC = x$ cm.
Then $ED = FC - FE - DC = (30 \text{ cm}) - (5 \text{ cm}) - (x \text{ cm}) = (25 - x)$ cm.
The height of the trapezoid is the length of $AF$, which is 14 cm.
Since the area of the trapezoid is 266 cm$^2$, then

$$\begin{aligned}
266 \text{ cm}^2 &= \frac{30 \text{ cm} + (25 - x) \text{ cm}}{2} \times (14 \text{ cm}) \\
266 &= \frac{55 - x}{2} \times 14 \\
532 &= (55 - x) \times 14 \\
38 &= 55 - x \\
x &= 55 - 38
\end{aligned}$$

and so $DE = x \text{ cm} = 17$ cm.

*Solution 2*
Let $DC = x$ cm.
Rectangle $ABCF$ has $AB = 30$ cm and $AF = 14$ cm, and so the area of $ABCF$ is $(30 \text{ cm}) \times (14 \text{ cm}) = 420 \text{ cm}^2$.
The area of $\triangle AFE$, which is right-angled at $F$, is

$$\tfrac{1}{2} \times AF \times FE = \tfrac{1}{2} \times (14 \text{ cm}) \times (5 \text{ cm}) = 35 \text{ cm}^2$$

The area of quadrilateral $ABDE$ is $266 \text{ cm}^2$.
The area of $\triangle BCD$, which is right-angled at $C$, is

$$\tfrac{1}{2} \times BC \times DC = \tfrac{1}{2} \times (14 \text{ cm}) \times (x \text{ cm}) = 7x \text{ cm}^2$$

Comparing the area of rectangle $ABCF$ to the combined areas of the pieces, we obtain

$$\begin{aligned} (35 \text{ cm}^2) + (266 \text{ cm}^2) + (7x \text{ cm}^2) &= 420 \text{ cm}^2 \\ 301 + 7x &= 420 \\ 7x &= 119 \\ x &= 17 \end{aligned}$$

Thus, the length of $DC$ is 17 cm.

ANSWER: (A)

17. Megan's car travels 100 m at $\frac{5}{4}$ m/s, and so takes $\dfrac{100 \text{ m}}{5/4 \text{ m/s}} = \dfrac{400}{5} \text{ s} = 80$ s.
Hana's car completes the 100 m in 5 s fewer, and so takes 75 s.
Thus, the average speed of Hana's car was $\dfrac{100 \text{ m}}{75 \text{ s}} = \dfrac{100}{75} \text{ m/s} = \dfrac{4}{3}$ m/s.

ANSWER: (C)

18. The number of bars taken from the boxes is $1 + 2 + 4 + 8 + 16 = 31$.
If these bars all had mass 100 g, their total mass would be 3100 g.
Since their total mass is 2920 g, they are $3100 \text{ g} - 2920 \text{ g} = 180$ g lighter.
Since all of the bars have a mass of 100 g or of 90 g, then it must be the case that 18 of the bars are each 10 g lighter (that is, have a mass of 90 g).
Thus, we want to write 18 as the sum of two of 1, 2, 4, 8, 16 in order to determine the boxes from which the 90 g bars were taken.
We note that $18 = 2 + 16$ and so the 90 g bars must have been taken from box $W$ and box $Z$.
Can you see why this is unique?

ANSWER: (B)

19. Since the average of $a$, $b$ and $c$ is 16, then $\dfrac{a+b+c}{3} = 16$ and so $a + b + c = 3 \times 16 = 48$.
Since the average of $c$, $d$ and $e$ is 26, then $\dfrac{c+d+e}{3} = 26$ and so $c + d + e = 3 \times 26 = 78$.
Since the average of $a$, $b$, $c$, $d$, and $e$ is 20, then $\dfrac{a+b+c+d+e}{5} = 20$.
Thus, $a + b + c + d + e = 5 \times 20 = 100$.
We note that

$$(a + b + c) + (c + d + e) = (a + b + c + d + e) + c$$

and so $48 + 78 = 100 + c$ which gives $c = 126 - 100 = 26$.

ANSWER: (D)

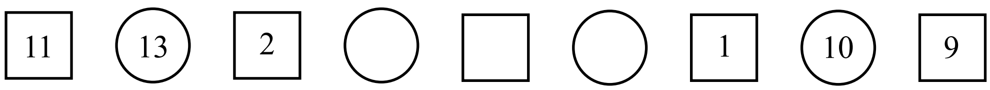
20. Each group of four jumps takes the grasshopper 1 cm to the east and 3 cm to the west, which is a net movement of 2 cm to the west, and 2 cm to the north and 4 cm to the south, which is a net movement of 2 cm to the south.
In other words, we can consider each group of four jumps, starting with the first, as resulting in a net movement of 2 cm to the west and 2 cm to the south.
We note that $158 = 2 \times 79$.
Thus, after 79 groups of four jumps, the grasshopper is $79 \times 2 = 158$ cm to the west and 158 cm to the south of its original position. (We need at least 79 groups of these because the grasshopper cannot be 158 cm to the south of its original position before the end of 79 such groups.)
The grasshopper has made $4 \times 79 = 316$ jumps so far.
After the 317th jump (1 cm to the east), the grasshopper is 157 cm west and 158 cm south of its original position.
After the 318th jump (2 cm to the north), the grasshopper is 157 cm west and 156 cm south of its original position.
After the 319th jump (3 cm to the west), the grasshopper is 160 cm west and 156 cm south of its original position.
After the 320th jump (4 cm to the south), the grasshopper is 160 cm west and 160 cm south of its original position.
After the 321st jump (1 cm to the east), the grasshopper is 159 cm west and 160 cm south of its original position.
After the 322nd jump (2 cm to the north), the grasshopper is 159 cm west and 158 cm south of its original position.
After the 323rd jump (3 cm to the west), the grasshopper is 162 cm west and 158 cm south of its original position, which is the desired position.
As the grasshopper continues jumping, each of its positions will always be at least 160 cm south of its original position, so this is the only time that it is at this position.
Therefore, $n = 323$. The sum of the squares of the digits of $n$ is $3^2 + 2^2 + 3^2 = 9 + 4 + 9 = 22$.

Answer: (A)

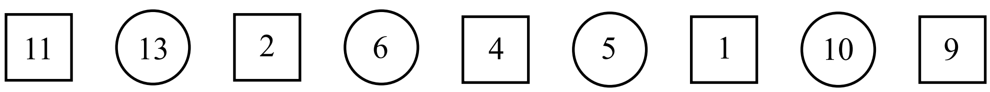
21. Since the line with equation $y = mx - 50$ passes through the point $(a, 0)$, then $0 = ma - 50$ or $ma = 50$.
Since $m$ and $a$ are positive integers whose product is 50, then $m$ and $a$ are a divisor pair of 50.
Therefore, the possible values of $m$ are the positive divisors of 50, which are 1, 2, 5, 10, 25, 50.
The sum of the possible values of $m$ is thus $1 + 2 + 5 + 10 + 25 + 50 = 93$.

Answer: 93

22. From the given information, if $a$ and $b$ are in two consecutive squares, then $a + b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a + b$ is larger than both $a$ and $b$.
This means that the largest integer in the list, which is 13, cannot be either $x$ or $y$ (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (because 13 is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x + y$ to be as large as possible, we would have $x$ and $y$ equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x + y$ is when $x = 9$ and $y = 11$. (We could also swap $x$ and $y$.)
Here, we could have $13 = 11 + 2$ and $10 = 9 + 1$, giving the following partial list:



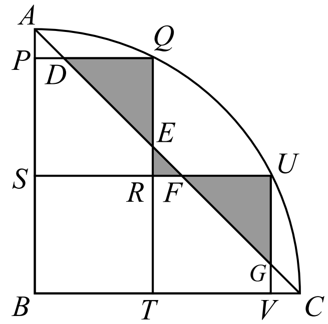

The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.




This tells us that the largest possible value of $x + y$ is 20.

Answer: 20

23. Since $AB$ and $BC$ are both radii of the circle, then $AB = BC$.
Since $ABC$ is a quarter-circle centred at $B$, then $\angle ABC = 90^\circ$.
Thus, $\triangle ABC$ is isosceles and right-angled, which means that $\angle BAC = \angle BCA = 45^\circ$.
We add some additional labels to the diagram:




We note that the angles between the straight lines at $P$, $Q$, $R$, and $U$ are all right angles.
Since $\angle PAD = \angle BAC = 45^\circ$, this means that

$$\angle PAD = \angle ADP = \angle QDE = \angle DEQ = \angle REF = \angle EFR = \angle UFG = \angle UGF = 45^\circ$$

This in turn means that each of $\triangle APD$, $\triangle DQE$, $\triangle ERF$, and $\triangle FUG$ is right-angled and isosceles.
Since the side length of each square is 10, then $BT = 10$ and $TQ = TR + RQ = 20$.
Since $\angle BTQ = 90^\circ$, then by the Pythagorean Theorem,

$$BQ = \sqrt{BT^2 + TQ^2} = \sqrt{10^2 + 20^2} = \sqrt{500}$$

We note that $\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}$.
Since $BQ$ is a radius of the circle, then $BQ = BA = BC = 10\sqrt{5}$.
Since $BP = TQ = 20$, then $AP = BA - BP = 10\sqrt{5} - 20$.
Thus, $PD = AP = 10\sqrt{5} - 20$.
Since $PQ = 10$, then $DQ = PQ - PD = 10 - (10\sqrt{5} - 20) = 30 - 10\sqrt{5}$.
Thus, $QE = DQ = 30 - 10\sqrt{5}$.
Since $PQ = QR$ and $DQ = QE$, then $PD = ER = 10\sqrt{5} - 20$.
Using similar reasoning, $ER = RF = 10\sqrt{5} - 20$ and $UF = UG = 30 - 10\sqrt{5}$.
The total area, $\mathcal{A}$, of the shaded regions equals the sum of the areas of $\triangle DQE$, $\triangle ERF$ and $\triangle FUG$.
Therefore,

$$\begin{aligned}
\mathcal{A} &= \tfrac{1}{2} \times DQ \times QE + \tfrac{1}{2} \times ER \times RF + \tfrac{1}{2} UF \times UG \\
&= \tfrac{1}{2}(30 - 10\sqrt{5})^2 + \tfrac{1}{2}(10\sqrt{5} - 20)^2 + \tfrac{1}{2}(30 - 10\sqrt{5})^2 \\
&= (30 - 10\sqrt{5})^2 + \tfrac{1}{2}(10\sqrt{5} - 20)^2 \\
&= (30^2 - 2 \times 30 \times 10\sqrt{5} + (10\sqrt{5})^2) + \tfrac{1}{2}((10\sqrt{5})^2 - 2 \times 10\sqrt{5} \times 20 + 20^2) \\
&= (900 - 600\sqrt{5} + 500) + \tfrac{1}{2}(500 - 400\sqrt{5} + 400) \\
&= 1400 - 600\sqrt{5} + 450 - 200\sqrt{5} \\
&= 1850 - 800\sqrt{5} \\
&\approx 61.14
\end{aligned}$$

and so the integer closest to $\mathcal{A}$ is 61.

Answer: 61

24. We want to determine the probability that Carina wins 3 games before she loses 2 games.
This means that she either wins 3 and loses 0, or wins 3 and loses 1.
If Carina wins her first three games, we do not need to consider the case of Carina losing her fourth game, because we can stop after she wins 3 games.
Putting this another way, once Carina has won her third game, the outcomes of any later games do not affect the probability because wins or losses at that stage will not affect the question that is being asked.
Using W to represent a win and L to represent a loss, the possible sequence of wins and losses that we need to examine are WWW, LWWW, WLWW, and WWLW.

In the case of WWW, the probabilities of the specific outcome in each of the three games are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{4}$, because the probability of a win after a win is $\frac{3}{4}$.
Threfore, the probability of WWW is $\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$.

In the case of LWWW, the probabilities of the specific outcome in each of the four games are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{3}{4}$, because the probability of a loss in the first game is $\frac{1}{2}$, the probability of a win after a loss is $\frac{1}{3}$, and the probability of a win after a win is $\frac{3}{4}$.
Threfore, the probability of LWWW is $\frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{96} = \frac{3}{32}$.

Using similar arguments, the probability of WLWW is $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{3} \times \frac{3}{4} = \frac{3}{96} = \frac{1}{32}$.
Here, we used the fact that the probability of a loss after a win is $1 - \frac{3}{4} = \frac{1}{4}$.

Finally, the probability of WWLW is $\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{3} = \frac{3}{96} = \frac{1}{32}$.

Therefore, the probability that Carina wins 3 games before she loses 2 games is $\frac{9}{32} + \frac{3}{32} + \frac{1}{32} + \frac{1}{32} = \frac{14}{32} = \frac{7}{16}$, which is in lowest terms.
The sum of the numerator and denominator of this fraction is 23.

Answer: 23

25. *Solution 1*
Let $N = AB\,0AB$ and let $t$ be the two-digit integer $AB$.
We note that $N = 1001t$, and that $1001 = 11 \cdot 91 = 11 \cdot 7 \cdot 13$.
Therefore, $N = t \cdot 7 \cdot 11 \cdot 13$.
We want to write $N$ as the product of 5 distinct odd integers, each greater than 2, and to count the number of sets $S$ of such odd integers whose product is $N$.
There are several situations to consider.
First, we look at possible sets $S$ that include the integers 7, 11 and 13 (which we know are divisors).
Second, we look at possible sets $S$ that include two of these integers and an odd multiple of the third.
Third, we rule out possible sets $S$ that include one of these integers and odd multiples of the second and third.
Fourth, we rule out possible sets $S$ that include the product of two or three of these integers and additional integers.

Case 1: $S = \{7, 11, 13, m, n\}$ where $m < n$ and $m, n \neq 7, 11, 13$

Here, $N = 7 \cdot 11 \cdot 13 \cdot m \cdot n = mn \cdot 1001$ and so $t = mn$. This tells us that $mn$ is less than 100.
If $m = 3$, then the possible values for $n$ are 5, 9, 15, 17, 19, 21, 23, 25, 27, 29, 31.
These give the following corresponding values of $mn$: 15, 27, 45, 51, 57, 63, 69, 75, 81, 87, 93.
Note that $n \neq 33$, since $m = 3$ and $n = 33$ gives $mn = 99$ which has two equal digits and so is not possible.
If $m = 5$, then the possible values for $n$ are 9, 15, 17, 19.

If $m \geq 9$, then $n \geq 15$ since the integers in $n$ are odd and distinct, and so $mn \geq 135$, which is not possible.
Therefore, in this case, there are 15 possible sets.

Case 2: $S = \{7q, 11, 13, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7q, 11, 13$

Here, we have $N = 7q \cdot 11 \cdot 13 \cdot m \cdot n$ and so $N = 1001 \cdot mnq$ which gives $t = mnq$.
Note that $mnq \leq 99$.
Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, then the possible values of $n$ are 5 and 9 since $m$ and $n$ are odd, greater than 2, and distinct. ($n = 7$ is not possible, since this would give the set $\{21, 11, 13, 3, 7\}$ which is already counted in Case 1 above.)
If $m \geq 5$, then $n \geq 7$ which gives $mn \geq 35$, which is not posible.
Suppose that $q = 5$. This means that $mn \leq \frac{99}{5} = 19\frac{4}{5}$.
If $m = 3$, then $n = 5$. There are no further possibilities when $q = 5$.
Since $mn \geq 3 \cdot 5 = 15$ and $mnq \leq 99$, then we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 3: $S = \{7, 11q, 13, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7, 11q, 13$

Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, then the possible values of $n$ are 5 and 9. (Note that $n \neq 7$.) We cannot have $n = 11$ as this would give $mnq = 99$ and a product of 99 099 which has equal digits $A$ and $B$.
We cannot have $m \geq 5$ since this gives $mn \geq 45$.
Suppose that $q = 5$. This means that $mn \leq \frac{99}{5}$.
If $m = 3$, then $n = 5$.
As in Case 2, we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 4: $S = \{7, 11, 13q, m, n\}$ where $m < n$ and $q > 1$ is odd and $m, n \neq 7, 11, 13q$

Suppose that $q = 3$. This means that $mn \leq 33$.
If $m = 3$, the possible values of $n$ are 5 and 9. (Again, $n \neq 11$ in this case.)
We cannot have $m \geq 5$ when $q = 3$ otherwise $mn \geq 45$.
If $q = 5$, we can have $m = 3$ and $n = 5$ but there are no other possibilities.
As in Cases 2 and 3, we cannot have $q \geq 7$.
Therefore, in this case, there are 3 possible sets.

Case 5: $S = \{7q, 11r, 13, m, n\}$ where $m < n$ and $q, r > 1$ are odd and $m, n \neq 7q, 11r, 13$

Here, $mnqr \leq 99$.
Since $q, r > 1$ are odd, then $qr \geq 9$ which means that $mn \leq 11$.
Since there do not exist two distinct odd integers greater than 1 with a product less than 15, there are no possible sets in this case.
A similarly argument rules out the products

$$N = 7q \cdot 11 \cdot 13r \cdot m \cdot n \qquad N = 7 \cdot 11q \cdot 13r \cdot m \cdot n \qquad N = 7q \cdot 11r \cdot 13s \cdot m \cdot n$$

where $q$, $r$, $s$ are odd integers greater than 1.

Case 6: $S = \{77, 13, m, n, \ell\}$ where $m < n < \ell$ and $m, n, \ell \neq 77, 13$

Note that $77 = 7 \cdot 11$ since we know that $N$ has divisors of 7 and 11.
Here, $mn\ell \leq 99$.
Since $mn\ell \geq 3 \cdot 5 \cdot 7 = 105$, there are no possible sets in this case, nor using $7 \cdot 143$ or $11 \cdot 91$ in the product or 1001 by itself or multiples of 77, 91 or 143.

Having considered all cases, there are $15 + 3 + 3 + 3 = 24$ possible sets.

*Solution 2*

We note first that $AB\,0AB = AB \cdot 1001$, and that $1001 = 11 \cdot 91 = 11 \cdot 7 \cdot 13$.
Therefore, $AB\,0AB = AB \cdot 7 \cdot 11 \cdot 13$.
Since $AB\,0AB$ is odd, then $B$ is odd.
Since $A \neq 0$ and $A \neq B$ and $B$ is odd, then we have the following possibilities for the two-digit integer $AB$:

$$13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41, 43, 45, 47, 49$$

$$51, 53, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97$$

If the integer $AB$ is a prime number, then $AB\,0AB$ cannot be written as the product of five different positive integers each greater than 2, since it would have at most four prime factors.
Using this information, we can eliminate many possibilities for $AB$ from our list to obtain the shorter list:

$$15, 21, 25, 27, 35, 39, 45, 49, 51, 57, 63, 65, 69, 75, 81, 85, 87, 91, 93, 95$$

Several of the integers in this shorter list are the product of two distinct prime numbers neither of which is equal to 7, 11 or 13. These integers are $15 = 3 \cdot 5$ and $51 = 3 \cdot 17$ and $57 = 3 \cdot 19$ and $69 = 3 \cdot 23$ and $85 = 5 \cdot 17$ and $87 = 3 \cdot 29$ and $93 = 3 \cdot 31$ and $95 = 5 \cdot 19$.
Thinking about each of these as $p \cdot q$ for some distinct prime numbers $p$ and $q$, we have $AB\,0AB = p \cdot q \cdot 7 \cdot 11 \cdot 13$.
To write $AB\,0AB$ as the product of five different positive odd integers greater each greater than 2, these five integers must be the five prime factors. For each of these 8 integers (15, 51, 57, 69, 85, 87, 93, 95), there is 1 set of five distinct odd integers, since the order of the integers does not matter. This is 8 sets so far.
This leaves the integers 21, 25, 27, 35, 39, 45, 49, 63, 65, 75, 81, 91.

Seven of these remaining integers are equal to the product of two prime numbers, which are either equal primes or at least one of which is equal to 7, 11 or 13. These products are $21 = 3 \cdot 7$ and $25 = 5 \cdot 5$ and $35 = 5 \cdot 7$ and $39 = 3 \cdot 13$ and $49 = 7 \cdot 7$ and $65 = 5 \cdot 13$ and $91 = 7 \cdot 13$.
In each case, $AB\,0AB$ can then be written as a product of 5 prime numbers, at least 2 of which are the same. These 5 prime numbers cannot be grouped to obtain five different odd integers, each larger than 1, since the 5 prime numbers include duplicates and if two of the primes are combined, we must include 1 in the set. Consider, for example, $21 = 3 \cdot 7$. Here, $21\,021 = 3 \cdot 7 \cdot 7 \cdot 11 \cdot 13$. There is no way to group these prime factors to obtain five different odd integers, each larger than 1.
Similary, $25\,025 = 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $91\,091 = 7 \cdot 13 \cdot 7 \cdot 11 \cdot 13$. The remaining three possibilities (35, 49 and 65) give similar situations.
This leaves the integers 27, 45, 63, 75, 81 to consider.

Consider $27\,027 = 3^3 \cdot 7 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product. Since there cannot be two 3's in the set, the only way to do this so that they are all different is $\{3, 9, 7, 11, 13\}$.
Consider $81\,081 = 3^4 \cdot 7 \cdot 11 \cdot 13$. There are 7 prime factors to distribute among the five odd integers that form the product.
Since there cannot be two 3s or two 9s in the set and there must be two powers of 3 in the set, there are four possibilities for the set $S$:

$$S = \{3, 27, 7, 11, 13\}, \{3, 9, 21, 11, 13\}, \{3, 9, 7, 33, 13\}, \{3, 9, 7, 11, 39\}$$

Consider $45\,045 = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product.

Since two of these prime factors are 3, they cannot each be an individual element of the set and so one of the 3s must always be combined with another prime giving the following possibilities:

$$S = \{9, 5, 7, 11, 13\}, \{3, 15, 7, 11, 13\}, \{3, 5, 21, 11, 13\}, \{3, 5, 7, 33, 13\}, \{3, 5, 7, 11, 39\}$$

Consider $75\,075 = 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.
Using a similar argument to that in the case of 45 045, we obtain

$$S = \{15, 5, 7, 11, 13\}, \{3, 25, 7, 11, 13\}, \{3, 5, 35, 11, 13\}, \{3, 5, 7, 55, 13\}, \{3, 5, 7, 11, 65\}$$

Finally, consider $63\,063 = 3^2 \cdot 7^2 \cdot 11 \cdot 13$. There are 6 prime factors to distribute among the five odd integers that form the product. Since we cannot have two 3s or two 7s in the product, the second 3 and the second 7 must be combined, and so there is only one set in this case, namely $S = \{3, 7, 21, 11, 13\}$.

We have determined that the total number of sets is thus $8 + 1 + 4 + 5 + 5 + 1 = 24$.

Answer: 24