The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2023 Canadian Team Mathematics Contest

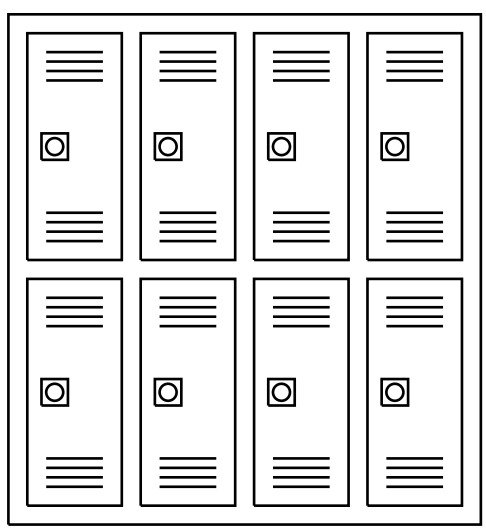
Individual Problems (45 minutes)

IMPORTANT NOTES:

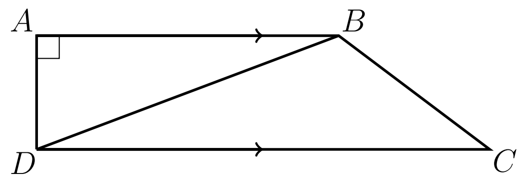
- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. Ingrid starts with n chocolates, while Jin and Brian each start with 0 chocolates. Ingrid gives one third of her chocolates to Jin. Jin gives 8 chocolates to Brian and then Jin eats half of her remaining chocolates. Jin now has 5 chocolates. What is the value of n?
2. For what value of k is $k\%$ of 25 equal to 20% of 30?
3. It is now 1:00 a.m. What time will it be 2023 minutes from now?
4. A group of eight students have lockers that are arranged as shown, in two rows of four lockers with one row directly on top of the other. The students are allowed to paint their lockers either blue or red according to two rules. The first rule is that there must be two blue lockers and two red lockers in each row. The second rule is that lockers in the same column must have different colours. How many ways are there for the students to paint their lockers according to the rules?

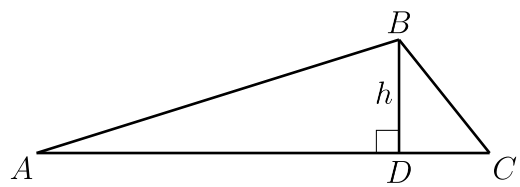


5. In trapezoid $ABCD$, $AB = 4$, $CD = 6$, $\angle DAB = 90°$, $\angle BCD = 45°$, and AB is parallel to CD. What is the length of BD?



6. A train is traveling from City A to City B. If the train travels at a speed of 80 km/h, it will arrive 24 minutes late. If it travels at a speed of 90 km/h, it will arrive 32 minutes early. At what speed in km/h should the train travel in order to arrive on time?

7. In $\triangle ABC$, $\tan\angle BCA = 1$ and $\tan\angle BAC = \frac{1}{7}$. The perimeter of $\triangle ABC$ is $24 + 18\sqrt{2}$. The altitude from B to AC has length h and intersects AC at D. What is the value of h?



8. A *Tim number* is a five-digit positive integer with the property that it is a multiple of 15, its hundreds digit is 3, and its tens digit is equal to the sum of its first (leftmost) three digits. How many Tim numbers are there?

9. The real numbers x, y and z satisfy both of the equations below:

$$4x + 7y + z = 11$$
$$3x + y + 5z = 15$$

Given that $x + y + z = \frac{p}{q}$ where p and q are positive integers and the fraction $\frac{p}{q}$ is in lowest terms, what is the value of $p - q$?

10. For every positive integer n, let $S_n = \{1, 2, 3, \ldots, n\}$; that is, S_n is the set of integers from 1 to n inclusive. There are 2^n subsets of S_n. If each subset has the same likelihood of being chosen, let $p(n)$ be the probability that a chosen subset does not contain two integers with a sum of $n + 1$.

For example, the subsets of S_2 are $\emptyset$ (the empty set), $\{1\}$, $\{2\}$, and $\{1, 2\}$. Of these four subsets, only $\{1, 2\}$ contains a pair of integers with a sum of $2 + 1 = 3$. The other three subsets do not contain such a pair, so $p(2) = \frac{3}{4}$.

What is the smallest *even* positive integer n for which $p(n) < \frac{1}{4}$?

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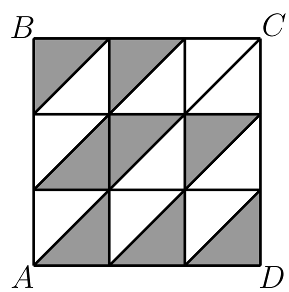
Relay Problem #1 (Seat a)

A square garden has an area of 9 square metres. The perimeter of the garden is N metres. What is the value of N?

Relay Problem #1 (Seat b)

Let t be TNYWR.
In the diagram, square $ABCD$ has side-length t. Two vertical lines and two horizontal lines divide square $ABCD$ into nine equal smaller squares, and each of these smaller squares is cut in half by a diagonal, as shown. Some of the triangular regions defined by these lines are shaded. What is the total area of the shaded parts of the square?



Relay Problem #1 (Seat c)

Let t be TNYWR.
If $t = n(n-1)(n+1) + n$, what is the value of n?

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Relay Problem #2 (Seat a)

How many integers n with $n > 0$ satisfy $\dfrac{1}{n+1} > \dfrac{4}{29}$?

Relay Problem #2 (Seat b)

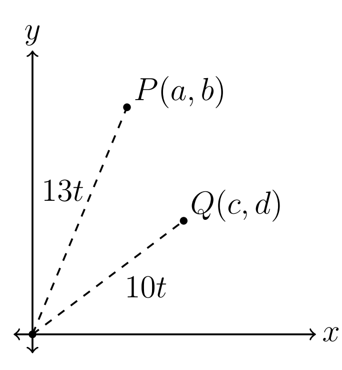
Let t be TNYWR.

A water tank initially contains x litres of water. Pablo adds $\dfrac{t}{2}$ litres of water to the tank, making the tank 20% full. After Pablo adds water to the tank, Chloe then adds $\dfrac{t^2}{4}$ litres of water to the tank, making the tank 50% full. What is the value of x?

Relay Problem #2 (Seat c)

Let t be TNYWR.

Point O is at the origin and points $P(a,b)$ and $Q(c,d)$ are in the first quadrant, as shown. The slope of OP is $\dfrac{12}{5}$ and the length of OP is $13t$. The slope of OQ is $\dfrac{3}{4}$ and the length of OQ is $10t$. What is $a+c$?



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Relay Problem #3 (Seat a)

Three of the positive divisors of 24 are 1, 8, and 24. What is the sum of all of the positive divisors of 24?

Relay Problem #3 (Seat b)

Let t be TNYWR.
The numbers a and b satisfy both of the following equations.

$$a - \frac{t}{6}b = 20$$
$$a - \frac{t}{5}b = -10$$

What is the value of b?

Relay Problem #3 (Seat c)

Let t be TNYWR.
The parabola with equation $y = ax^2 + bx + c$ passes through $(4, 0)$, $\left(\frac{t}{3}, 0\right)$, and $(0, 60)$. What is the value of a?

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Team Problems (45 minutes)

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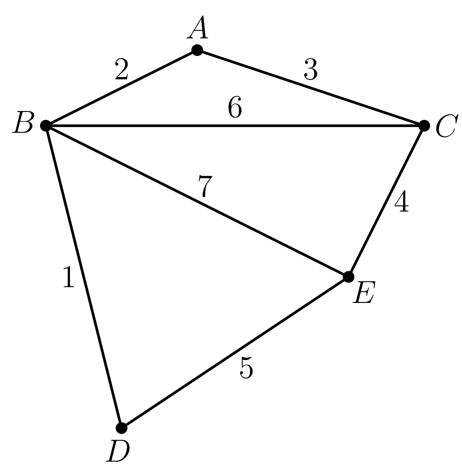
PROBLEMS:

1. A fish and chips truck sells three kinds of fish: cod, haddock, and halibut. During a music festival, the truck sold 220 pieces of fish, of which 40% were haddock and 40% were halibut. How many pieces of cod did the truck sell?
2. If $\dfrac{x}{2} - 5 = 9$, what is the value of $\sqrt{7x}$?
3. Point A has coordinates $(-13, -23)$ and point B has coordinates $(-33, -43)$. What is the slope of a line that is perpendicular to the line segment AB?
4. What is the integer equal to $\sqrt{\dfrac{119^2 - 17^2}{119 - 17} - 10^2}$?
5. If $p + q + r = 70$, $p = 2q$, and $q = 3r$, what is the value of p?
6. How many ordered triples (a, b, c) of integers satisfy $1 \leq a < b < c \leq 10$ and $b - a = c - b$?
7. The distinct prime factors of 18 are 2 and 3. What is the sum of all of the distinct prime factors of 4446?
8. Using the diagram below, a seven digit integer can be created as follows: trace a path that uses each line segment exactly once and use the labels on the line segments as digits. For example, the path that goes from C to A to B to C to E to B to D and finally to E gives the positive integer 3264715. What is the largest possible positive integer that can be created in this way?



9. Alheri has three hoses. Water flows out of each hose at the same constant rate. Using all three hoses, she starts to fill a swimming pool at 6:00 a.m. and calculates that the pool will be full at exactly 6:00 p.m. on the same day. At 11:00 a.m., one of the hoses unexpectedly stops working. Assuming water still flows out of the other two hoses at the same rate as before, at what time will the pool be full?

10. The lines with the two equations below intersect at the point $(2, -3)$.

$$\begin{aligned}(a^2+1)x - 2by &= 4\\ (1-a)x + by &= 9\end{aligned}$$

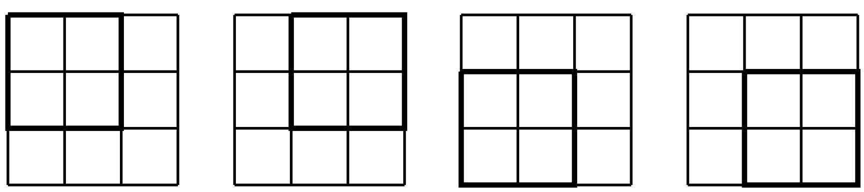
What are the possible ordered pairs (a, b)?

11. What is the integer equal to $2023^4 - (2022)(2024)(1 + 2023^2)$?

12. For how many integers x is the expression $\dfrac{\sqrt{75-x}}{\sqrt{x-25}}$ equal to an integer?

13. A positive integer is called *mystical* if it has at least two digits and every pair of two consecutive digits, read from left to right, forms a perfect square. For example, 364 is a mystical integer because 36 and 64 are both perfect squares, but 325 is not mystical because 32 is not a perfect square. What is the largest mystical integer?

14. In a 3×3 grid, there are four 2×2 *subgrids*, each of which is bordered by a thick line in one of the four grids below.



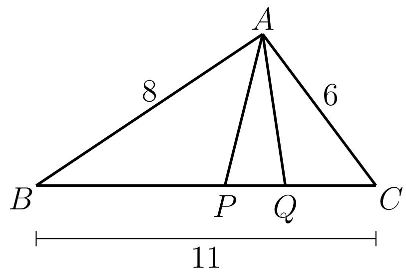
Kasun wants to place an integer from 1 to 4 inclusive in each cell of a 3×3 grid so that every 2×2 subgrid contains each integer exactly once. For example, the grid below and on the left satisfies the condition, but the grid below and on the right does not. In how many ways can Kasun place integers in the grid so that they satisfy the condition?

1	2	3
3	4	1
1	2	3

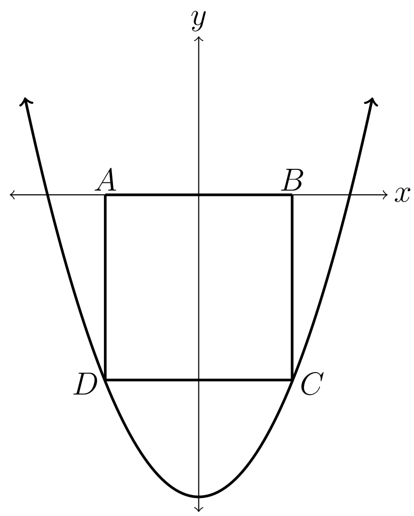
1	3	2
2	4	1
1	3	3

15. There are exactly three real numbers x for which $\left(x - \dfrac{5}{x}\right)$ is the reciprocal of $(x-4)$. What is the sum of these three real numbers?

16. In $\triangle ABC$, $AB = 8$, $BC = 11$, and $AC = 6$. The points P and Q are on BC such that $\triangle PBA$ and $\triangle QAC$ are each similar to $\triangle ABC$. What is the length of PQ?



17. Square $ABCD$ has A and B on the x-axis and C and D below the x-axis on the parabola with equation $y = x^2 - 4$. What is the area of $ABCD$?



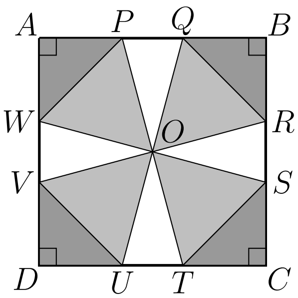
18. Let $a = \log_4 9$ and $b = 108 \log_3 8$. What is the integer equal to $\sqrt{ab}$?

19. Jolene and Tia are playing a two-player game at a carnival. In one bin, there are five red balls numbered 5, 10, 15, 20 and 25. In another bin, there are 25 green balls numbered 1 through 25. In the first stage of the game, Jolene chooses one of the red balls at random. Next, the carnival worker removes the green ball with the same number as the ball Jolene chose. Tia then chooses one of the 24 remaining green balls at random.

    Jolene and Tia win if the number on the ball chosen by Tia is a multiple of 3. What is the probability that they will win?

20. The positive integer d has the property that each of 468, 636, and 867 has the same remainder, r, when divided by d. What is the largest possible value of $d + r$?

21. Square $ABCD$ has centre O. Points P and Q are on AB, R and S are on BC, T and U are on CD, and V and W are on AD, as shown, so that $\triangle APW$, $\triangle BRQ$, $\triangle CTS$, and $\triangle DVU$ are isosceles and $\triangle POW$, $\triangle ROQ$, $\triangle TOS$, and $\triangle VOU$ are equilateral. What is the ratio of the area of $\triangle PQO$ to that of $\triangle BRQ$?



22. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference 3 and $a_1 = 1$. The sequence $b_1, b_2, b_3, \ldots$ is an arithmetic sequence with common difference 10 and $b_1 = 2$. What is the smallest integer larger than 2023 that appears in both sequences?

23. In $\triangle ABC$, $AB = 10$ and $\sin 3A + 3\sin C = 4$. What is the length of the altitude from C to AB?

24. The real numbers x, y, and z satisfy both of the equations below.

$$x + y + z = 2$$
$$xy + yz + xz = 0$$

Let a be the minimum possible value of z and b be the maximum possible value of z. What is the value of $b - a$?

25. Let $f(x)$ be a function with the property that $f(x) + f\left(\frac{x-1}{3x-2}\right) = x$ for all real numbers x other than $\frac{2}{3}$. What is the sum $f(0) + f(1) + f(2)$?