The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Canadian Senior Mathematics Contest

Wednesday, November 15, 2023
(in North America and South America)

Thursday, November 16, 2023
(outside of North America and South America)

Solutions

Part A

1. Since $p+q$ is odd (because 31 is odd) and p and q are integers, then one of p and q is even and the other is odd. (If both were even or both were odd, their sum would be even.)
   Since p and q are both prime numbers and one of them is even, then one of them must be 2, since 2 is the only even prime number.
   Since their sum is 31, the second number must be 29, which is prime.
   Therefore, $pq = 2 \cdot 29 = 58$.

   Answer: 58

2. The integers between 100 to 999, inclusive, are exactly the three-digit positive integers.
   Consider three-digit integers of the form abc where the digit a is even, the digit b is even, and the digit c is odd.
   There are 4 possibilities for a: 2, 4, 6, 8. (We note that a cannot equal 0.)
   There are 5 possibilities for b: 0, 2, 4, 6, 8.
   There are 5 possibilities for c: 1, 3, 5, 7, 9.
   Each choice of digits from these lists gives a distinct integer that satisfies the conditions.
   Therefore, the number of such integers is $4 \cdot 5 \cdot 5 = 100$.

   Answer: 100

3. *Solution 1*
   Since the distance from $(0,0)$ to (x,y) is 17, then $x^2+y^2=17^2$.
   Since the distance from $(16,0)$ to (x,y) is 17, then $(x-16)^2+y^2=17^2$.
   Subtracting the second of these equations from the first, we obtain $x^2-(x-16)^2=0$ which gives $x^2-(x^2-32x+256)=0$ and so $32x=256$ or $x=8$.
   Since $x=8$ and $x^2+y^2=17^2$, then $64+y^2=289$ which gives $y^2=225$, from which we get $y=15$ or $y=-15$.
   Therefore, the two possible pairs of coordinates for P are $(8,15)$ and $(8,-15)$.

   Solution 2
   The point P is equidistant from O and A since $OP=PA=17$.
   Suppose that M is the midpoint of OA.
   Since O has coordinates $(0,0)$ and A has coordinates $(16,0)$, then M has coordinates $(8,0)$.
   Since $OP=PA$, then $\triangle OPA$ is isosceles.
   This means that median PM in $\triangle OPA$ is also an altitude; in other words, PM is perpendicular to OA.
   Since OA is horizontal, PM is vertical, and so P lies on the vertical line with equation $x=8$.
   Since $OM=8$ and $OP=17$ and $\triangle PMO$ is right-angled at M, then by the Pythagorean Theorem, $PM=\sqrt{OP^2-OM^2}=\sqrt{17^2-8^2}=\sqrt{225}=15$.
   Since PM is vertical and M is on the x-axis, then P is a distance of 15 units vertically from the x-axis.
   Since P has x-coordinate 8 and is 15 units away from the x-axis, then the two possible pairs of coordinates for P are $(8,15)$ and $(8,-15)$.

   Answer: $(8,15)$, $(8,-15)$

4. The store sold x shirts for \$10 each, y water bottles for \$5 each, and z chocolate bars for \$1 each.
Since the total revenue was \$120, then $10x + 5y + z = 120$.
Since $z = 120 - 10x - 5y$ and each term on the right side is a multiple of 5, then z is a multiple of 5.
Set $z = 5t$ for some integer $t > 0$.
This gives $10x + 5y + 5t = 120$. Dividing by 5, we obtain $2x + y + t = 24$.
Since $x > 0$ and x is an integer, then $x \geq 1$.
Since $y > 0$ and $t > 0$, then $y + t \geq 2$ (since y and t are integers).
This means that $2x = 24 - y - t \leq 22$ and so $x \leq 11$.
If $x = 1$, then $y + t = 22$. There are 21 pairs (y, t) that satisfy this equation, namely the pairs $(y, t) = (1, 21), (2, 20), (3, 19), \ldots, (20, 2), (21, 1)$.
If $x = 2$, then $y + t = 20$. There are 19 pairs (y, t) that satisfy this equation, namely the pairs $(y, t) = (1, 19), (2, 18), (3, 17), \ldots, (18, 2), (19, 1)$.
For each value of x with $1 \leq x \leq 11$, we obtain $y + t = 24 - 2x$.
Since $y \geq 1$, then $t \leq 23 - 2x$.
Since $t \geq 1$, then $y \leq 23 - 2x$.
In other words, $1 \leq y \leq 23 - 2x$ and $1 \leq t \leq 23 - 2x$.
Furthermore, picking any integer y satisfying $1 \leq y \leq 23 - 2x$ gives a positive value of t, and so there are $23 - 2x$ pairs (y, t) that are solutions.
Therefore, as x ranges from 1 to 11, there are

$$21 + 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1$$

pairs (y, t), which means that there are this number of triples (x, y, z).
This sum can be re-written as

$$21 + (19 + 1) + (17 + 3) + (15 + 5) + (13 + 7) + (11 + 9)$$

or $21 + 5 \cdot 20$, which means that the number of triples is 121.

ANSWER: 121

5. We consider $r^2 - r(p + 6) + p^2 + 5p + 6 = 0$ to be a quadratic equation in r with two coefficients that depend on the variable p.
For this quadratic equation to have real numbers r that are solutions, its discriminant, Δ, must be greater than or equal to 0. A non-negative discriminant does not guarantee integer solutions, but may help us narrow the search.
By definition,

$$\begin{aligned} \Delta &= (-(p + 6))^2 - 4 \cdot 1 \cdot (p^2 + 5p + 6) \\ &= p^2 + 12p + 36 - 4p^2 - 20p - 24 \\ &= -3p^2 - 8p + 12 \end{aligned}$$

Thus, we would like to find all integer values of p for which $-3p^2 - 8p + 12 \geq 0$. The set of integers p that satisfy this inequality are the only possible values of p which could be part of a solution pair (r, p) of integers. We can visualize the left side of this inequality as a parabola opening downwards, so there will be a finite range of values of p for which this is true.
By the quadratic formula, the solutions to the equation $-3p^2 - 8p + 12 = 0$ are

$$p = \frac{8 \pm \sqrt{8^2 - 4(-3)(12)}}{2(-3)} = \frac{8 \pm \sqrt{208}}{-6} \approx 1.07, -3.74$$

Since the roots of the equation $-3p^2 - 8p + 12 = 0$ are approximately 1.07 and -3.74, then the integers p for which $-3p^2 - 8p + 12 \geq 0$ are $p = -3, -2, -1, 0, 1$. (These values of p are the only integers between the real solutions 1.07 and -3.74.)
It is these values of p for which there are possibly integer values of r that work.
We try them one by one:

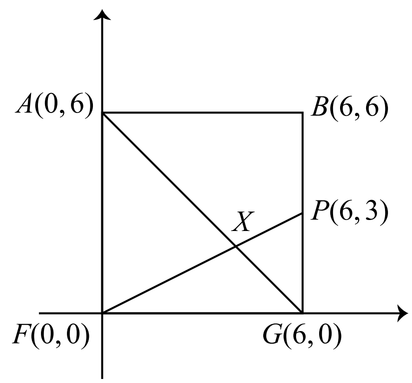
- When $p = 1$, the original equation becomes $r^2 - 7r + 12 = 0$, which gives $(r-3)(r-4) = 0$, and so $r = 3$ or $r = 4$.
- When $p = 0$, the original equation becomes $r^2 - 6r + 6 = 0$. Using the quadratic formula, we can check that this equation does not have integer solutions.
- When $p = -1$, the original equation becomes $r^2 - 5r + 2 = 0$. Using the quadratic formula, we can check that this equation does not have integer solutions.
- When $p = -2$, the original equation becomes $r^2 - 4r = 0$, which factors as $r(r - 4) = 0$, and so $r = 0$ or $r = 4$.
- When $p = -3$, the original equation becomes $r^2 - 3r = 0$, which factors as $r(r - 3) = 0$, and so $r = 0$ or $r = 3$.

Therefore, the pairs of integers that solve the equation are

$$(r, p) = (3, 1), (4, 1), (0, -2), (4, -2), (0, -3), (3, -3)$$

Answer: $(3, 1), (4, 1), (0, -2), (4, -2), (0, -3), (3, -3)$

6. We start by determining the heights above the bottom of the cube of the points of intersection of the edges of the pyramids.
For example, consider square $AFGB$ and edges AG and FP. We call their point of intersection X.
We assign coordinates to the various points using the fact that the edge length of the cube is 6: $F(0,0)$, $G(6,0)$, $B(6,6)$, $A(0,6)$, $P(6,3)$ (P is the midpoint of BG).



Line segment AG has slope -1, and so has equation $y = -x + 6$.
Line segment FP has slope $\frac{3}{6} = \frac{1}{2}$ and so has equation $y = \frac{1}{2}x$.
To find the coordinates of X, we equate expressions in y to obtain $-x + 6 = \frac{1}{2}x$ which gives $\frac{3}{2}x = 6$ or $x = 4$, and so $y = -4 + 6 = 2$.
Therefore, point X is a height of 2 above square $EFGH$.
Using a similar argument, the point of intersection between PH and GC is 2 units above square $EFGH$.
To see why the point of intersection of GD and PE is also 2 units above $EFGH$, we note that rectangle $DEGB$ has a height of 6 (like square $AFGB$) and a width of $6\sqrt{2}$. As a result, we can think of obtaining rectangle $DEGB$ by stretching square $AFGB$ horizontally by a factor of $\sqrt{2}$. This horizontal stretch will not raise or lower the point of intersection between GD and PE and so this point is also two units above $EFGH$.
Now, imagine drawing a plane through the three points of intersection of the edges of the pyramids.
Since each of these points is 2 units above $EFGH$, this plane must be horizontal and will also intersect BG 2 units above G, forming a square. (The points of intersection form a square because every horizontal cross-section of both pyramids is a square.) This square has side length 2 because the x-coordinate of X was 4, which is 2 units from BG in that coordinate system.
This square divides the common three-dimensional region into two square-based pyramids.
One of these pyramids points upwards and has fifth vertex P. This pyramid has a square base with edge length 2 and a height of $3 - 2 = 1$, since P is 3 units above G and the base of the pyramid is 2 units above G.
The other pyramid points downwards and has fifth vertex G. This pyramid has a square base with edge length 2 and a height of 2.
Thus, the volume of the region is $\frac{1}{3} \cdot 2^2 \cdot 1 + \frac{1}{3} \cdot 2^2 \cdot 2 = \frac{4}{3} + \frac{8}{3} = 4$.

Answer: 4

Part B

1. (a) Since AB is parallel to DC and AD is perpendicular to both AB and DC, then the area of trapezoid $ABCD$ is equal to $\frac{1}{2} \cdot AD \cdot (AB + DC)$ or $\frac{1}{2} \cdot 10 \cdot (7 + 17) = 120$.

   Alternatively, we could separate trapezoid $ABCD$ into rectangle $ABFD$ and right-angled triangle $\triangle BFC$.
   We note that $ABFD$ is a rectangle since it has three right angles.
   Rectangle $ABFD$ is 7 by 10 and so has area 70.
   $\triangle BFC$ has BF perpendicular to FC and has $BF = AD = 10$.
   Also, $FC = DC - DF = DC - AB = 17 - 7 = 10$.
   Thus, the area of $\triangle BFC$ is $\frac{1}{2} \cdot FC \cdot BF = \frac{1}{2} \cdot 10 \cdot 10 = 50$.

   This means that the area of trapezoid $ABCD$ is $70 + 50 = 120$.

   (b) Since PQ is parallel to DC, then $\angle BQP = \angle BCF$.
   We note that $ABFD$ is a rectangle since it has three right angles. This means that $BF = AD = 10$ and $DF = AB = 7$.
   In $\triangle BCF$, we have $BF = 10$ and $FC = DC - DF = 17 - 7 = 10$.
   Therefore, $\triangle BCF$ has $BF = FC$, which means that it is right-angled and isosceles.
   Therefore, $\angle BCF = 45^\circ$ and so $\angle BQP = 45^\circ$.

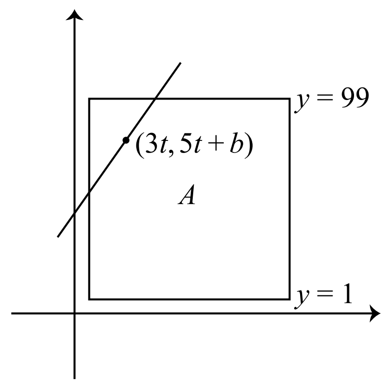
   (c) Since PQ is parallel to AB and AP and BT are perpendicular to AB, then $ABTP$ is a rectangle.
   Thus, $AP = BT$ and $PT = AB = 7$.
   Since $PT = 7$, then $TQ = PQ - PT = x - 7$.
   Since $\angle BQT = 45^\circ$ and $\angle BTQ = 90^\circ$, then $\triangle BTQ$ is right-angled and isosceles.
   Therefore, $BT = TQ = x - 7$.
   Finally, $AP = BT = x - 7$.

   (d) Suppose that $PQ = x$.
   In this case, trapezoid $ABQP$ has parallel sides $AB = 7$ and $PQ = x$, and height $AP = x - 7$.
   The areas of trapezoid $ABQP$ and trapezoid $PQCD$ are equal exactly when the area of trapezoid $ABQP$ is equal to half of the area of trapezoid $ABCD$.
   Thus, the areas of $ABQP$ and $PQCD$ are equal exactly when $\frac{1}{2}(x - 7)(x + 7) = \frac{1}{2} \cdot 120$, which gives $x^2 - 49 = 120$ or $x^2 = 169$.
   Since $x > 0$, then $PQ = x = 13$.

   Alternatively, we could note that trapezoid $PQCD$ has parallel sides $PQ = x$ and $DC = 17$, and height $PD = AD - AP = 10 - (x - 7) = 17 - x$.
   Thus, the area of trapezoid $ABQP$ and the area of trapezoid $PQCD$ are equal exactly when $\frac{1}{2}(x-7)(x+7) = \frac{1}{2}(17-x)(x+17)$, which gives $x^2-49 = 17^2-x^2$ or $x^2-49 = 289-x^2$ and so $2x^2 = 338$ or $x^2 = 169$.
   Since $x > 0$, then $PQ = x = 13$.

2. (a) The lattice points inside the region A are precisely those lattice points whose coordinates (r, s) satisfy $1 \le r \le 99$ and $1 \le s \le 99$.
Each point on the line with equation $y = 2x+5$ is of the form $(a, 2a+5)$ and so each lattice point on the line with equation $y = 2x + 5$ is of the form $(a, 2a + 5)$ for some integer a.
For such a lattice point to lie in region A, we need $1 \le a \le 99$ and $1 \le 2a + 5 \le 99$.
The second pair of inequalities is equivalent to $-4 \le 2a \le 94$ and thus to $-2 \le a \le 47$.
Since we need both $1 \le a \le 99$ and $-2 \le a \le 47$ to be true, we have $1 \le a \le 47$.
Since there are 47 integers a in this range, then there are 47 lattice points in the region A and on the line with equation $y = 2x + 5$.
These are the points $(1, 7), (2, 9), (3, 11), \ldots, (47, 99)$.

(b) Consider a lattice point (r, s) that lies on the line with equation $y = \frac{5}{3}x + b$.
In this case, we must have $s = \frac{5}{3}r + b$ and so $\frac{5}{3}r = s - b$.
Since s and b are both integers, then $\frac{5}{3}r$ is an integer.
Since r is an integer and $\frac{5}{3}r$ is an integer, then r is a multiple of 3.
We write $r = 3t$ for some integer t which means that $s = \frac{5}{3} \cdot 3t + b = 5t + b$.
Thus, the lattice point (r, s) can be re-written as $(3t, 5t + b)$.
For $(3t, 5t + b)$ to lie within A, we need $1 \le 3t \le 99$.



Since t is an integer, this means that $1 \le t \le 33$.
When $b = 0$, these points are the points of the form $(3t, 5t)$; these lie within A when $1 \le t \le 19$. In other words, there are 19 points in A when $b = 0$, which means that the greatest possible value of b is at least 0.
We note that $5t + b$ is increasing as t increases.
When $b \ge 0$ and $t \ge 1$, we have $5t+b \ge 5$ and so if any points lie within A, then the point with $t = 1$ must lie within A. This means that for at least 15 of the points $(3t, 5t + b)$ to lie within A, the points corresponding to $t = 1, 2, \ldots, 14, 15$ must all lie within A.
Since $5t + b$ is increasing, the largest value of b should correspond to the largest value of $5(15) + b$ that does not exceed 99.
When $b = 24$, we note that the points for $t = 1, 2, \ldots, 14, 15$ are

$$(r, s) = (3, 29), (6, 34), \ldots, (42, 94), (45, 99)$$

which means that exactly 15 points lie within A.
We note that if $b \ge 25$ and $t \ge 15$, then $5t + b \ge 100$ and so the point $(3t, 5t + b)$ is not within A; in other words, if $b \ge 25$, there are fewer than 15 points on the line that lie within A.
Therefore, $b = 24$ is indeed the largest possible value of b that satisfies the given requirements.

(c) Consider a line with equation $y = mx + 1$ for some value of m.
Regardless of the value of m, the point $(0, 1)$ lies on this line. This point is not in the region A, but is right next to it.
Consider the line with equation $y = \frac{3}{7}x + 1$ (that is, $m = \frac{3}{7}$).
The point $(7, 4)$ is a lattice point in A that lies on this line.
This means that $m = \frac{3}{7}$ cannot be in the final range of values, and so n cannot be greater than $\frac{3}{7}$.
Consider the points on the line with equation $y = mx + 1$ with x-coordinates from 1 to 99, inclusive. These are the points

$$(1, m + 1), (2, 2m + 1), (3, 3m + 1), \ldots, (98, 98m + 1), (99, 99m + 1)$$

Since $m < n \leq \frac{3}{7}$, then $99m + 1 < 99 \cdot \frac{3}{7} + 1 < 99$ and so each of these 99 points are in the region A.
This means that we need to ensure that none of $m+1, 2m+1, 3m+1, \ldots, 98m+1, 99m+1$ is an integer.
In other words, we want to determine the greatest possible real number n for which none of $m + 1, 2m + 1, 3m + 1, \ldots, 98m + 1, 99m + 1$ is an integer whenever $\frac{2}{7} < m < n$.
Since real numbers s and $s+1$ are either both integers or both not integers, then we want to determine the greatest possible real number n for which none of $m, 2m, 3m, \ldots, 98m, 99m$ is an integer whenever $\frac{2}{7} < m < n$.
The fact that none of $m, 2m, 3m, \ldots, 98m, 99m$ can be an integer is equivalent to saying that m is not equal to a rational number of the form $\dfrac{c}{d}$ where c is an integer and d is equal to one of $1, 2, 3, \ldots, 98, 99$.
This means that the value of n that we want is the largest real number n with the property that there are no rational numbers $m = \dfrac{c}{d}$ with c and d integers and $1 \leq d \leq 99$ in the interval $\dfrac{2}{7} < m < n$.
Let s be the smallest rational number of the form $\dfrac{c}{d}$ with c and d integers and $1 \leq d \leq 99$ that is greater than $\dfrac{2}{7}$.
Then it must be the case that $n = s$.
To see why this is true, we note that s has the property that there are no rational numbers m with the above restrictions between $\dfrac{2}{7}$ and s by the definition of s, and also that any number larger than s does not have this property because s would be between it and $\frac{2}{7}$.
Therefore, $n = s$.
This means that we need to determine the smallest rational number of the form $\dfrac{c}{d}$ with c and d integers and $1 \leq d \leq 99$ that is greater than $\dfrac{2}{7}$.
To do this, we minimize the value of $\dfrac{c}{d} - \dfrac{2}{7} = \dfrac{7c - 2d}{7d}$ subject to the conditions that c and d are positive integers with $1 \leq d \leq 99$ and that $\dfrac{c}{d} - \dfrac{2}{7} = \dfrac{7c - 2d}{7d} > 0$, which also means that $7c - 2d > 0$.
When $d = 99$, we are minimizing $\dfrac{7c - 198}{693}$ which is the smallest possible when $c = 29$, giving a difference of $\dfrac{5}{693}$.
When $d = 98$, we are minimizing $\dfrac{7c - 196}{686}$ which is the smallest possible when $c = 29$,

giving a difference of $\frac{7}{686}$.

When $d = 97$, we are minimizing $\frac{7c - 194}{679}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{2}{679}$.

When $d = 96$, we are minimizing $\frac{7c - 192}{672}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{4}{672}$.

When $d = 95$, we are minimizing $\frac{7c - 190}{665}$ which is the smallest possible when $c = 28$, giving a difference of $\frac{6}{665}$.

When $d = 94$, we are minimizing $\frac{7c - 188}{658}$ which is the smallest possible when $c = 27$, giving a difference of $\frac{1}{658}$.

We can check that $\frac{1}{658}$ is smaller than any of $\frac{5}{693}, \frac{7}{686}, \frac{2}{679}, \frac{4}{672}, \frac{6}{665}$.

Furthermore, if $d < 94$, then since $\frac{7c - 2d}{7d} \geq \frac{1}{7d} > \frac{1}{658}$ (noting that $7c - 2d \geq 1$) and so every other difference will be greater than $\frac{1}{658}$.

This means that $\frac{27}{94}$ is the smallest of this set of rational numbers, which means that $n = \frac{27}{94}$.

3. (a) Working with x in degrees

We know that $\sin\theta = 1$ exactly when $\theta = 90^\circ + 360^\circ k$ for some integer k.

Therefore, $\sin\left(\frac{x}{5}\right) = 1$ exactly when $\frac{x}{5} = 90^\circ + 360^\circ k_1$ for some integer k_1 which gives $x = 450^\circ + 1800^\circ k_1$.

Also, $\sin\left(\frac{x}{9}\right) = 1$ exactly when $\frac{x}{9} = 90^\circ + 360^\circ k_2$ for some integer k_2 which gives $x = 810^\circ + 3240^\circ k_2$.

Equating expressions for x, we obtain

$$\begin{aligned} 450^\circ + 1800^\circ k_1 &= 810^\circ + 3240^\circ k_2 \\ 1800k_1 - 3240k_2 &= 360 \\ 5k_1 - 9k_2 &= 1 \end{aligned}$$

One solution to this equation is $k_1 = 2$ and $k_2 = 1$.

These give $x = 4050^\circ$. We note that $\frac{x}{5} = 810^\circ$ and $\frac{x}{9} = 450^\circ$; both of these angles have a sine of 1.

Working with x in radians

We know that $\sin\theta = 1$ exactly when $\theta = \frac{\pi}{2} + 2\pi k$ for some integer k.

Therefore, $\sin\left(\frac{x}{5}\right) = 1$ exactly when $\frac{x}{5} = \frac{\pi}{2} + 2\pi k_1$ for some integer k_1 which gives $x = \frac{5\pi}{2} + 10\pi k_1$.

Also, $\sin\left(\frac{x}{9}\right) = 1$ exactly when $\frac{x}{9} = \frac{\pi}{2} + 2\pi k_2$ for some integer k_2 which gives $x = \frac{9\pi}{2} + 18\pi k_2$.

Equating expressions for x, we obtain

$$\begin{aligned} \frac{5\pi}{2} + 10\pi k_1 &= \frac{9\pi}{2} + 18\pi k_2 \\ 10\pi k_1 - 18\pi k_2 &= 2\pi \\ 5k_1 - 9k_2 &= 1 \end{aligned}$$

One solution to this equation is $k_1 = 2$ and $k_2 = 1$.

These give $x = \frac{45\pi}{2}$. We note that $\frac{x}{5} = \frac{9\pi}{2}$ and $\frac{x}{9} = \frac{5\pi}{2}$; both of these angles have a sine of 1.

Therefore, one solution is $x = 4050°$ (in degrees) or $x = \frac{45\pi}{2}$ (in radians).

(b) Suppose that M and N are positive integers.

We work towards determining conditions on M and N for which there is or is not an angle x with $\sin\left(\frac{x}{M}\right) + \sin\left(\frac{x}{N}\right) = 2$.

Since $-1 \leq \sin\theta \leq 1$ for all angles θ, then the equation $\sin\left(\frac{x}{M}\right) + \sin\left(\frac{x}{N}\right) = 2$ is equivalent to the pair of equations $\sin\left(\frac{x}{M}\right) = \sin\left(\frac{x}{N}\right) = 1$. (Putting this another way, there must be an angle x which makes both sines 1 simultaneously.)

As in (a), the equation $\sin\left(\frac{x}{M}\right) = 1$ is equivalent to the statement that $\frac{x}{M} = 90° + 360°r$ or $\frac{x}{M} = \frac{\pi}{2} + 2\pi r$ for some integer r. (We will carry equations in degrees and in radians simultaneously for a time.)

These equations are equivalent to saying $x = 90°M + 360°rM$ or $x = \frac{M\pi}{2} + 2\pi rM$ for some integer r.

Similarly, the equation $\sin\left(\frac{x}{N}\right) = 1$ is equivalent to saying $x = 90°N + 360°sN$ or $x = \frac{N\pi}{2} + 2\pi sN$ for some integer s.

Since x is common, then we can equate values of x to say that if such an x exists, then $90°M + 360°rM = 90°N + 360°sN$ or $\frac{M\pi}{2} + 2\pi rM = \frac{N\pi}{2} + 2\pi sN$.

It is also true that if these equations are true, then the existence of an angle x that satisfies, say, $x = 90°M + 360°rM$ then guarantees the fact that the same angle x satisfies $x = 90°N + 360°sN$.

In other words, the existence of an angle x is equivalent to the existence of integers r and s for which $90°M + 360°rM = 90°N + 360°sN$ or $\frac{M\pi}{2} + 2\pi rM = \frac{N\pi}{2} + 2\pi sN$.

Dividing the first equation throughout by $90°$ and the second equation throughout by $\frac{\pi}{2}$ gives us the same resulting equation, namely $M + 4rM = N + 4sN$. Thus, we can not concern ourselves with using degrees or radians for the rest of this part.

At this stage, we know that there is an angle x with the desired property precisely when there are integers r and s for which $M + 4rM = N + 4sN$.

Suppose that $M = 2^a c$ and $N = 2^b d$ for some integers a, b, c, d with $a \geq 0$, $b \geq 0$, c odd, and d odd. Here, we are writing M and N as the product of a power of 2 and their "odd part".
Suppose that $a \neq b$; without loss of generality, assume that $a > b$.
Then, the following equations are equivalent:

$$\begin{aligned} M + 4rM &= N + 4sN \\ 2^a c + 4r \cdot 2^a c &= 2^b d + 4s \cdot 2^b d \\ 2^{a-b} c + 2^{2+a-b} rc &= d + 4sd \\ 2^{a-b} c + 2^{2+a-b} rc - 4sd &= d \end{aligned}$$

Since the right side of this equation is an odd integer and the left side is an even integer regardless of the choice of r and s, there are no integers r and s for which this is true.
Thus, if M and N do not contain the same number of factors of 2, there is no angle x that satisfies the initial equation.

To see this in another way, we return to the equation $M + 4rM = N + 4sN$, factor both sides to obtain $M(1 + 4r) = N(1 + 4s)$ which gives the equivalent equation $\dfrac{M}{N} = \dfrac{1 + 4s}{1 + 4r}$.
If integers r and s exist that satisfy this equation, then $\dfrac{M}{N}$ can be written as a ratio of odd integers and so M and N must contain the same number of factors of 2.
Putting this another way, if M and N do not contain the same number of factors of 2, then integers r and s do not exist and so the initial equation has no solutions.

To complete (b), we need to demonstrate the existence of a sequence $n_1, n_2, \ldots, n_{100}$ of positive integers for which $\sin\left(\dfrac{x}{n_i}\right) + \sin\left(\dfrac{x}{n_j}\right) \neq 2$ for all angles x and for all pairs $1 \leq i < j \leq 100$.
Suppose that $n_i = 2^i$ for $1 \leq i \leq 100$.
In other words, the sequence $n_1, n_2, \ldots, n_{100}$ is the sequence $2^1, 2^2, \ldots, 2^{100}$.
No pair of numbers from the sequence $n_1, n_2, \ldots, n_{100}$ contains the same number of factors of 2, and so there is no angle x that makes $\sin\left(\dfrac{x}{n_i}\right) + \sin\left(\dfrac{x}{n_j}\right) = 2$ for any i and j with $1 \leq i < j \leq 100$.
Therefore, the sequence $n_i = 2^i$ for $1 \leq i \leq 100$ has the desired property.

(c) Suppose that M and N are positive integers for which there is an angle x that satisfies the equation $\sin\left(\dfrac{x}{M}\right) + \sin\left(\dfrac{x}{N}\right) = 2$.
From (b), we know that M and N must contain the same number of factors of 2.
Again, suppose that $M = 2^a c$ and $N = 2^a d$ for some integers a, c, d with $a \geq 0$, c odd, and d odd.
Then, continuing from earlier work, the following equations are equivalent:

$$\begin{aligned} M + 4rM &= N + 4sN \\ 2^a c + 4r \cdot 2^a c &= 2^a d + 4s \cdot 2^a d \\ c + 4rc &= d + 4sd \\ c - d &= -4rc + 4sd \end{aligned}$$

Since the right side is a multiple of 4, then the left side must also be a multiple of 4 and so c and d have the same remainder when divided by 4.

(Using a more advanced result from number theory, it turns out that if $c - d$ is divisible by 4, then this equation will always have a solution for the integers r and s, but we do not need this precise fact.)

Suppose that m_1, m_2, …, m_{100} is a list of 100 distinct positive integers with the property that, for each integer $i = 1, 2, \ldots, 99$, there is an angle x_i that satisfies the equation $\sin\left(\dfrac{x_i}{m_i}\right) + \sin\left(\dfrac{x_i}{m_{i+1}}\right) = 2$.
Suppose further that $m_1 = 6$.
Since $m_1 = 2^1 \cdot 3$ and there is an angle x_1 with $\sin\left(\dfrac{x_1}{m_1}\right) + \sin\left(\dfrac{x_1}{m_2}\right) = 2$, then from above, $m_2 = 2^1 \cdot c_2$ for some positive integer c_2 that is 3 more than a multiple of 4 (that is, c_2 has the same remainder upon division by 4 as 3 does).
Similarly, each integer in the list m_1, m_2, …, m_{100} can be written as $m_i = 2c_i$ where c_i is a positive integer that is 3 more than a multiple of 4.
Define $t = \dfrac{3\pi}{2^{100}} \cdot m_1 m_2 \cdots m_{100}$.
Then $\dfrac{t}{m_i} = \dfrac{3\pi}{2 \cdot 2^{99}(2c_i)}(2c_1)(2c_2)\cdots(2c_{100}) = \dfrac{\pi}{2} \cdot \dfrac{3c_1c_2\cdots c_{100}}{c_i}$.
In other words, $\dfrac{t}{m_i}$ is equal to $\dfrac{\pi}{2}$ times the product of 100 integers each of which is 3 more than a multiple of 4. (Note that the numerator of the last fraction includes 101 such integers and the denominator includes 1.)
The product of two integers each of which is 3 more than a multiple of 4 is equal to an integer that is 1 more than a multiple of 4. This is because if y and z are integers, then

$$(4y + 3)(4z + 3) = 16yz + 12y + 12z + 9 = 4(4yz + 3y + 3z + 2) + 1$$

Also, the product of two integers each of which is 1 more than a multiple of 4 is equal to an integer that is 1 more than a multiple of 4. This is because if y and z are integers, then

$$(4y + 1)(4z + 1) = 16yz + 4y + 4z + 1 = 4(4yz + y + z) + 1$$

Thus, the product of 100 integers each of which is 3 more than a multiple of 4 is equal to the product of 50 integers each of which is 1 more than a multiple of 4, which is equal to an integer that is one more than a multiple of 4.
Therefore, $\dfrac{t}{m_i}$ is equal to $\dfrac{\pi}{2}$ times an integer that is 1 more than a multiple of 4, and so $\sin\left(\dfrac{t}{m_i}\right) = 1$, and so

$$\sin\left(\frac{t}{m_1}\right) + \sin\left(\frac{t}{m_2}\right) + \cdots + \sin\left(\frac{t}{m_{100}}\right) = 100$$

as required.
Therefore, for every such sequence $m_1, m_2, \ldots, m_{100}$, there does exist an angle t with the required property.