The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Canadian Intermediate Mathematics Contest

Wednesday, November 15, 2023
(in North America and South America)

Thursday, November 16, 2023
(outside of North America and South America)

Solutions

Part A

1. The bus leaves at exactly 7:43 a.m., which means that after the bus has travelled for exactly 17 minutes, it will be 8:00 a.m.
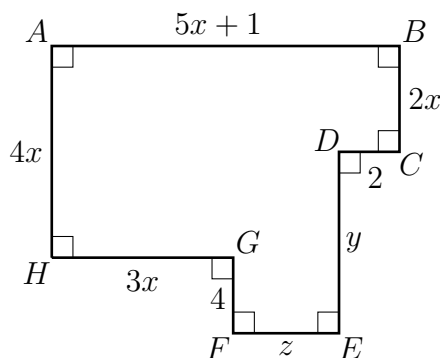   The bus arrives at its destination at exactly 8:22 a.m., which is exactly 22 minutes after 8:00 a.m.
   Therefore, the number of minutes the trip took was $17 + 22 = 39$.

   ANSWER: 39

2. Given that $a \blacklozenge 4 = 18$, we have $\dfrac{2a}{4} = 18$ or $\dfrac{a}{2} = 18$, and so $a = 36$.

   ANSWER: 36

3. We label the vertices of the figure by the letters A through H as shown.

   From the information given in the questions, $AB = 5x + 1$, $BC = 2x$, $CD = 2$, $FG = 4$, $GH = 3x$, and $AH = 4x$. We will set $DE = y$ and $EF = z$.
   Because of the fact that two line segments meet at a right angle if they meet at all, the sum of the lengths of the horizontal edges CD, EF, and GH must equal the length AB.
   Therefore, $2 + z + 3x = 5x + 1$, from which we obtain $z = 5x + 1 - (3x + 2) = 2x - 1$.
   By similar reasoning, $AH + FG = BC + DE$, so $4x + 4 = 2x + y$, from which we obtain $y = 4x + 4 - 2x = 2x + 4$.
   Using that $y = 2x+4$ and $z = 2x-1$, we can add the side lengths to get the following expression for the perimeter:

   $$(5x + 1) + 2x + 2 + (2x + 4) + (2x - 1) + 4 + 3x + 4x$$

   which can be simplified to get $18x + 10$.
   The perimeter is given to be 82, so $18x + 10 = 82$, which means $18x = 72$ or $x = 4$.

   ANSWER: 4

4. Since $3 \times 3 \times 3 = 27$, the larger cube is made by arranging the 27 smaller cubes in three layers so that each layer consists of 9 small cubes arranged in a 3×3 square.
   This means each face of the larger cube is made up of $3 \times 3 = 9$ faces of small cubes.
   A cube has six faces, so the surface of the cube is made up of $6 \times 9 = 54$ faces of small cubes.
   Of the 27 small cubes, 1 is completely concealed inside the larger cube, so none of its faces contribute to the surface of the larger cube.

The remaining 26 small cubes each contribute 1, 2, or 3 of their faces to the surface of the larger cube.

There are 6 small cubes that contribute 1 face to the surface of the larger cube, 12 that contribute 2 faces, and 8 that contribute 3 faces.

Observe that $6 + 12 + 8 = 26$, so this accounts for the remaining small cubes.

The 6 small cubes that contribute 1 face are in the centres of the faces of the larger cube.

The 12 small cubes that contribute 2 faces are on the edges of the larger cube.

The 8 small cubes that contribute 3 faces are on the corners of the larger cube.

Observe that $6 \times 1 + 12 \times 2 + 8 \times 3 = 54$, which is the correct total number of faces of small cubes that we expect on the surface of the larger cube.

The goal is to minimize the sum of the integers showing on the surface of the larger cube. This means we want to position each small cube so that the sum of the integers on its visible faces is as small as possible.

Each small cube has a face with the integer 1 on it. This means the 6 small cubes that each contribute 1 face to the surface of the larger cube can be positioned so that the number on this face is 1.

The 12 small cubes that contribute 2 faces to the surface of the cube contribute 2 faces that share an edge.

The smallest possible total of the integers on 2 faces of a small cube is $1 + 2 = 3$.

The faces showing 1 and 2 also share an edge (as indicated in the net given in the problem), so this means these 12 small cubes can be oriented so that they each contribute a total of $1 + 2 = 3$ to the total on the surface of the larger cube.

Each of the 8 small cubes that contribute three faces to the surface of the larger cube contribute 3 faces that meet at a vertex.

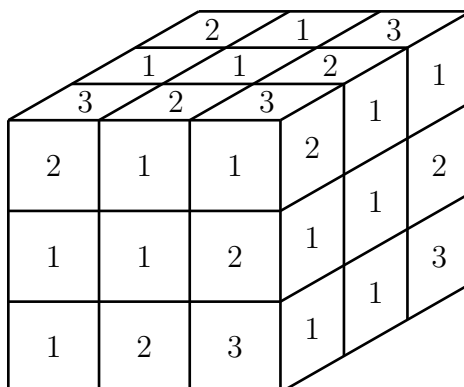The smallest possible sum of the integers on 3 faces of a small cube is $1 + 2 + 3 = 6$.

The faces that have the integers 1, 2, and 3 on them meet at a vertex.

This means these 8 cubes can be oriented so that they each contribute $1 + 2 + 3 = 6$ to the total on the surface of the larger cube.
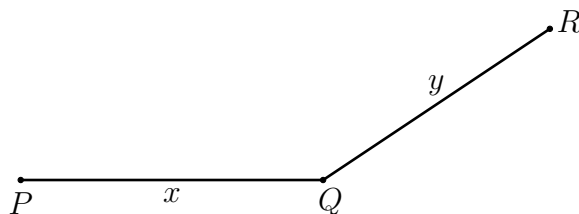
By minimizing the total for each of the 26 cubes that contributes to the surface of the larger cube, we minimize the overall total. Using the arguments above, the minimum possible total of the integers on the surface of the larger cube is

$$6 \times 1 + 12 \times 3 + 8 \times 6 = 90$$

Below is an image of what the larger cube could look like when the small cubes are arranged to minimize the total of the visible integers.

ANSWER: 90

5. Let the distance between P and Q be x km and the distance between Q and R be y km as shown in the diagram below.

The total distance travelled by the hiker was $(2x + 2y)$ km.

The total distance walked on flat ground was $2x$ km and this was done at a speed of 4 km/h.

Therefore, the total time the hiker spent walking on flat ground was $\dfrac{2x \text{ km}}{4 \text{ km/h}} = \dfrac{x}{2}$ h.

The total distance walked uphill was y km and this was done at a speed of 3 km/h.

Therefore, the total time spent walking uphill was $\dfrac{y}{3}$ h.

The total distance walked downhill was y km and this was done at a speed of 6 km/h.

Therefore, the total time spent walking downhill was $\dfrac{y}{6}$ h.

Since the hiker left point P at 1:00 p.m. and returned to point P at 8:00 p.m., the total time for the trip, including the hour spent at the viewing platform, was 7 hours.

Of these 7 hours, $7 - 1 = 6$ of them were spent walking.

The total times spent walking on flat ground, uphill, and downhill were $\dfrac{x}{2}$ h, $\dfrac{y}{3}$ h, and $\dfrac{y}{6}$ h.

Therefore, we have
$$\frac{x}{2} + \frac{y}{3} + \frac{y}{6} = 6$$

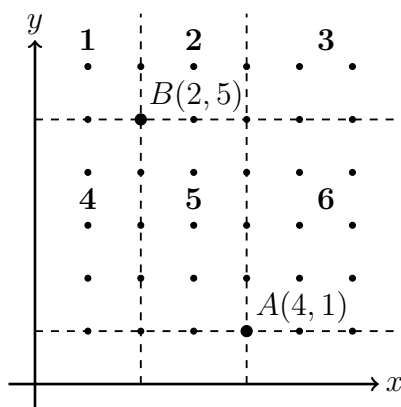Combining the left side into a single fraction, we get $\dfrac{3x + 2y + y}{6} = 6$ or $\dfrac{3x + 3y}{6} = 6$.

This simplifies to $\dfrac{x + y}{2} = 6$ or $x + y = 12$, and after multiplying both sides of this equation by 2, we get $2(x + y) = 24$ or $2x + 2y = 24$.

From earlier, the total distance travelled by the hiker was $(2x + 2y)$ km, so the answer is 24 km.

ANSWER: 24 km

6. In the diagram below, a dashed vertical line and a dashed horizontal line has been drawn through each of A and B.

These four lines, together with the y-axis, divide the set of points with positive integer coordinates into six regions. These regions are labelled **1** through **6** as shown in the diagram.

We will use the convention that a point on the horizontal boundary between two regions is considered to be in the top region, and a point on the vertical boundary between two regions is considered to be in the rightmost region.

This means, for example, that point B is in Region **2** and the point $(4, 3)$ is in Region **6**.

There are three other regions between the x-axis and the line with equation $y = 1$. However, following the convention stated above, there are no points with positive integer coordinates in any of these three regions.

For each of these six regions, we will examine the values of d_C for points C in that region. When we refer to the "positive difference" between two values, we mean the larger value minus the smaller value. We will allow the "positive difference" to be 0 if values are equal.

First, we make some general observations about how d_C can be computed.

The length of the shortest path from A to C is equal to the sum of the positive difference between the x-coordinates of A and C and the positive difference between the y-coordinates of A and C.

Suppose the positive difference between the x-coordinates is h and the positive difference between the y-coordinates is v.

Then there must be at least h horizontal moves of length 1 and v vertical moves of length 1 in any path from A to C, so the length of the shortest path is at least $h + v$.

However, a path of length $h + v$ is always possible because we can always move horizontally from A by h units (in the appropriate direction) and then move vertically by v units.

For the example in the problem statement, the coordinates of C are $(3, 4)$ and the coordinates of A are $(4, 1)$, and the length of the shortest path between the two is $(4 - 3) + (4 - 1) = 4$.

Similarly, the length of the shortest path from C to B is equal to the sum of the positive difference between the x-coordinates of C and B and the positive difference between the y-coordinates of C and B.

Region **1**: If $C(x, y)$ is in Region **1**, then $x = 1$ and $y \geq 5$. By the discussion above, the length of the shortest path from A to C is $(4 - 1) + (y - 1) = 2 + y$.

The length of the shortest path from C to B is $(2 - 1) + (y - 5) = -4 + y$.

Adding these lengths, we get that $d_C = (2 + y) + (-4 + y) = 2y - 2$ when C is in Region **1**.

For the point $C(1, 5)$, we thus have $d_C = 2(5) - 2 = 8$.

For the point $C(1, 6)$, we have $d_C = 2(6) - 2 = 10$.

Continuing in this way, if C and E are points in Region **1** with C one unit above E, then d_C is exactly two more than d_E.

Therefore, if C is in Region **1**, then d_C is an even integer that is at least 8. Moreover, for every even integer $N \geq 8$, there is exactly one point C in Region **1** such that $d_C = N$.

Region **2**: If $C(x, y)$ is in Region **2**, then $x = 2$ or $x = 3$ and $y \geq 5$.

The length of the shortest path from A to C is $(4 - x) + (y - 1) = 3 - x + y$.

The length of the shortest path from C to B is $(x - 2) + (y - 5) = -7 + x + y$.

In general, when $C(x, y)$ is in Region **2**, $d_C = (3 - x + y) + (-7 + x + y) = -4 + 2y$, which does not depend on the value of x.

For both of the points $C(2, 5)$ and $C(3, 5)$, we have $d_C = -4 + 2(5) = 6$.

For the points $C(2, 6)$ and $C(3, 6)$, $d_C = 8$.

By reasoning similar to that which was used when examining Region **1**, we see that for every even integer $N \geq 6$, there are exactly two points C in Region **2** such that $d_C = N$.

Region **6**: If $C(x, y)$ is in Region **6**, then $x \geq 4$ and y is equal to one of 1, 2, 3, or 4.

Using similar calculations to the previous cases, one can check that $d_C = 2x - 2$, which does not depend on y.

Similar to the observations for Regions **1** and **2**, the smallest possible value of d_C is when $x = 4$, which gives $d_C = 2(4) - 2 = 6$.

As well, if x increases by 1, then d_C increases by 2.

Therefore, for every even positive integer $N \geq 6$, there are exactly 4 (1 for each possible value of y) points C in Region **6** such that $d_C = N$.

Finally, we will examine Region **3**. We will not need to carefully examine Region **4** or Region **5** to answer the question. This will be explained after Region **3** is examined.

Region **3**: If $C(x, y)$ is in Region **3**, then $x \geq 4$ and $y \geq 5$.

In this case, it can be checked that $d_C = 2x + 2y - 12$.

The smallest possible value of d_C is when x and y are both as small as possible. This occurs when $(x, y) = (4, 5)$, which gives $d_C = 2(4) + 2(5) - 12 = 6$.

If exactly one of x or y increases by 1 (and the other does not change), then d_C increases by 2.

Thus, for $N = 8$, there are 2 points C for which $d_C = N$. These are the points one unit above $(4, 5)$ and one unit to the right of $(4, 5)$, which are $(4, 6)$ and $(5, 5)$, respectively.

To find points C with $d_C = 10$, we can move two units up from $(4, 5)$, or move one unit up *and* one unit to the right of $(4, 5)$, or move two units to the right of $(4, 5)$.

This corresponds to the 3 points $(6, 5)$, $(5, 6)$, and $(4, 7)$.

Thus, there are 3 points C in Region **3** with $d_C = 10$.

Continuing in this way, if k is a positive integer, the points C in Region **3** that satisfy $d_C = 6 + 2k$ are

- $(4 + k, 5)$,
- $(4 + (k - 1), 5 + 1)$,
- $(4 + (k - 2), 5 + 2)$,
- $\vdots$
- $(4 + 2, 5 + (k - 2))$,
- $(4 + 1, 5 + (k - 1))$, and
- $(4, 5 + k)$

of which there are exactly $k + 1$.

We now suppose N has the property that there are 2023 points C for which $d_C = N$.

There are only 12 points in total in Regions **4** and **5**.

Regions **1**, **2**, and **6** contain at most 1, 2, and 4, points C, respectively, that have $d_C = N$.

Therefore, Region **3** must contain at least $2023 - 12 - 1 - 2 - 4 = 2004$ of the points.

From above, we have that for each positive integer k, there are $k + 1$ points C in Region **3** with

the property that $d_C = 6 + 2k$.

We need $k + 1$ to be at least 2004, which means k must be at least 2003, so $N = d_C$ is at least $6 + 2(2003) = 4012$.

It is easily checked that no points C in Region **4** or Region **5** have d_C as large as 4012, so this means every point C with $d_C = N$ is in Region **1**, Region **2**, Region **3**, or Region **6**.

Moreover, since $N \geq 4012$, we can be sure that there is exactly one point C in Region **1** with $d_C = N$, there are exactly two points C in Region **2** with $d_C = N$, and exactly four points C in Region **6** with $d_C = N$.

Therefore, there are exactly $2023 - 1 - 2 - 4 = 2016$ points C in Region **3** with $d_C = N$.

This means $k + 1 = 2016$ or $k = 2015$, so $d_C = 6 + 2(2015) = 4036$.

ANSWER: 4036

Part B

1. (a) Since $ABCD$ is as square with side length $AB = 10$ cm, its area is $(10 \text{ cm})^2 = 100 \text{ cm}^2$.
Since $EFGH$ is a square with side length $EF = 8$ cm, is area is $(8 \text{ cm})^2 = 64 \text{ cm}^2$.
The shaded region is the region inside $ABCD$ that lies outside square $EFGH$, so its area is the difference between the two areas, or $100 \text{ cm}^2 - 64 \text{ cm}^2 = 36 \text{ cm}^2$.

   (b) The area of $ABCD$ is $(13 \text{ cm})^2 = 169 \text{ cm}^2$ and the area of the shaded region is 120 cm^2.
The area of $EFGH$ is $169 \text{ cm}^2 - 120 \text{ cm}^2 = 49 \text{ cm}^2$.
Since $EFGH$ is a square, its side length is $\sqrt{49 \text{ cm}^2} = 7$ cm.

   (c) The area of the shaded region is given to be $\dfrac{3}{4}$ of the total area of $ABCD$, so the area of $EFGH$ is $1 - \dfrac{3}{4} = \dfrac{1}{4}$ of the area of $ABCD$.
The area of $ABCD$ is $(18 \text{ cm})^2 = 324 \text{ cm}^2$.
The area of $EFGH$ is $\dfrac{1}{4}$ of the area of $ABCD$, so the area of $EFGH$ is $\dfrac{324}{4} \text{ cm}^2 = 81 \text{ cm}^2$.
Since $EFGH$ is a square, its side length is $\sqrt{81 \text{ cm}^2} = 9$ cm.

   (d) The area of $ABCD$ is $a^2 \text{ cm}^2$ and the area of $EFGH$ is $b^2 \text{ cm}^2$.
The given condition meanss that $a^2 - b^2 = \dfrac{64}{100}a^2$, which can be rearranged to $\dfrac{36}{100}a^2 = b^2$.
Taking square roots of both sides and noting that $a > 0$ and $b > 0$, we have $\sqrt{\dfrac{36}{100}a^2} = \sqrt{b^2}$ or $\dfrac{6}{10}a = b$, and so $\dfrac{3}{5}a = b$.
Multiplying both sides of this equation by $\dfrac{5}{3}$ gives $a = \dfrac{5}{3}b$.
Dividing both sides of this equation by b gives $\dfrac{a}{b} = \dfrac{5}{3}$.

2. (a) The RD sum of 4281 is $4281 + 1824 = 6105$.

   (b) The integer with digits $ABCD$ is equal to $1000A + 100B + 10C + D$.
The integer with digits $DCBA$ is equal to $1000D + 100C + 10B + A$.
The RD sum of $ABCD$ is

$$\begin{aligned} ABCD + DCBA &= (1000A + 100B + 10C + D) + (1000D + 100C + 10B + A) \\ &= 1001A + 110B + 110C + 1001D && \text{(group like terms)} \\ &= 1001(A + D) + 110(B + C) && \text{(factor)} \end{aligned}$$

and so $m = 1001$ and $n = 110$.

   (c) Suppose the RD sum of $ABCD$ is 3883.
From part (b), we have that $1001(A + D) + 110(B + C) = 3883$.
If $A + D$ were 4 or greater, then $1001(A + D)$ would be at least 4004, which is larger than 3883.
Therefore, $A + D$ is no larger than 3.
Since A and D are both nonzero digits, they are each at least 1, so $A \geq 1$ and $D \geq 1$, which implies $A + D \geq 1 + 1 = 2$.
This means $A + D = 2$ or $A + D = 3$.
If $A + D = 2$, then $1001(A + D) = 2002$.
Substituting this into $1001(A + D) + 110(B + C) = 3883$, we get $2002 + 110(B + C) = 3883$ or $110(B + C) = 1881$.

We also know that $B + C$ is an integer, so $110(B + C)$ is a multiple of 10, so it cannot be equal to 1881 which is not a multiple of 10.

Therefore, it is not possible that $A + D = 2$, so we must have $A + D = 3$.

Using that $A + D = 3$, we get $1001(A + D) = 3003$, so

$$110(B + C) = 3883 - 1001(A + D) = 3883 - 3003 = 880$$

Dividing $110(B + C) = 880$ on both sides by 110, we conclude that $B + C = 8$.

Since A and D are positive digits that have a sum of 3, we must have $A = 1$ and $D = 2$ or $A = 2$ and $D = 1$.

Since B and C are digits that have a sum of 8 but there is no restriction that they must be positive, we have that (B, C) can be any of the pairs $(0, 8)$, $(1, 7)$, $(2, 6)$, $(3, 5)$, $(4, 4)$, $(5, 3)$, $(6, 2)$, $(7, 1)$, or $(8, 0)$.

There are 2 possibilities for the values of A and D, and 9 possibilities for the values of B and C, Therefore, there are $2 \times 9 = 18$ integers with an RD sum of 3883. They are listed below.

$$1082, 1172, 1262, 1352, 1442, 1532, 1622, 1712, 1802$$

$$2081, 2171, 2261, 2351, 2441, 2531, 2621, 2711, 2801$$

(d) Suppose that n is a four-digit integer and that $ABCD$ is an integer with RD sum equal to n.

From part (b), this means $1001(A + D) + 110(B + C) = n$. Note that A and D are both non-zero.

Suppose there is some other integer, $EFGH$, with an RD sum equal to n.

Again by part (b), this means $1001(E + H) + 110(F + G) = n$.

Therefore, we have $1001(A + D) + 110(B + C) = 1001(E + H) + 110(F + G)$.

For convenience, set $w = A + D$, $x = B + C$, $y = E + H$, and $z = F + G$.

The equation above simplifies to $1001w + 110x = 1001y + 110z$.

Rearranging this equation, we get $1001w - 1001y = 110z - 110x$.

After factoring, we have $1001(w - y) = 110(z - x)$, and after dividing both sides of this equation, we get $91(w - y) = 10(z - x)$.

Observe that $91 = 7 \times 13$.

The quantities $w - y$ and $z - x$ are both integers, which means $91(w - y)$ and $10(z - x)$ are both integers.

Moreover, since $91(w - y)$ is a multiple of 13, we must also have that $10(z - x)$ is a multiple of 13.

The integer 13 is prime and 10 does not have a factor of 13, so $z - x$ must have a factor of 13.

By similar reasoning, $z - x$ must have a factor of 7, which means $z - x$ is a multiple of $7 \times 13 = 91$.

Now recall that each of x and z is the sum of two digits.

This means that each of x and z is an integer from 0 to 18 inclusive.

The difference between two integers that are at least 0 and at most 18 must lie between $0 - 18 = -18$ and $18 - 0 = 18$.

We have shown that $z - x$ is a multiple of 91 that lies between -18 and 18.

The only such integer is 0, which shows that $z - x = 0$ or $z = x$.

Since $z - x = 0$, $10(z - x) = 0$, which means $91(w - y) = 0$.

Therefore, $w - y = 0$ as well, so $w = y$.

We have shown that if $ABCD$ and $EFGH$ have the same RD sum, then $A + D = E + H$ and $B + C = F + G$.

In other words, if n is the RD sum of a four digit integer, then the values of $A + D$ and $B + C$ are uniquely determined.

This means, to count the number of four-digit RD sums, we can count the number of pairs (w, x) with the property that $1001w + 110x \leq 9999$, subject to the condition that w and x are the sum of digits, with w being the sum of positive digits.

Thus, the answer to the question is the number of pairs (w, x) with $2 \leq w \leq 18$ and $0 \leq x \leq 18$ that satisfy $1001w + 110x \leq 9999$.

If $w = 2$, then $1001w + 110x = 2002 + 110x$, so we want $2002 + 110x \leq 9999$ or $110x \leq 9999 - 2002$ or $110x \leq 7997$.

The largest that x can be is 18, so the largest that $110x$ can be is $18 \times 110 = 1980$, which is less than 7997.

Therefore, if $w = 2$, then x can be any integer from 0 through 18.

Thus, we get 19 pairs (w, x) when $w = 2$.

If $w = 3$, then $1001w + 110x = 3003 + 110x$, which means we need $110x \leq 6996$.

As with the case when $w = 2$, every possible value of x from 0 through 18 satisfies this inequality, so we get 19 pairs in this case as well.

If $w = 4$, similar reasoning implies that $110x \leq 5995$, so we again get 19 pairs.

Continuing this reasoning, we find that we get 19 pairs for each of $w = 5$, $w = 6$, $w = 7$, and $w = 8$.

If $w = 9$, then we must have that $9009 + 110x \leq 9999$, so $110x \leq 9999 - 9009$ or $110x \leq 990$. It is not difficult to see that x can be any integer from 0 through 9 inclusive, but if $x \geq 10$, then $110x \leq 990$ is not true. Therefore, we get 10 pairs in this case.

If $w \geq 10$, then $1001w$ is at least 10010, so there is no way to choose x so that $1001w + 110x$ has four digits.

Therefore, there are no more cases to consider, and the number of possible four-digit RD sums is

$$19 + 19 + 19 + 19 + 19 + 19 + 19 + 10 = 143$$

3. (a) The positive multiples of 7 that are no larger than $7^2 = 49$ are

$$7, 14, 21, 28, 35, 42, 49$$

None of these integers is in an earlier row, so they are exactly the integers in Row 7.
The positive multiples of 8 that are no larger than $8^2 = 64$ are

$$8, 16, 24, 32, 40, 48, 56, 64$$

Of these, 8, 16, and 24 are in earlier rows, so Row 8 is $32, 40, 48, 56, 64$.
The positive multiples of 9 that are no larger than $9^2 = 81$ are

$$9, 18, 27, 36, 45, 54, 63, 72, 81$$

Of these, 9, 18, and 36 are in earlier rows, so Row 9 is $27, 45, 54, 63, 72, 81$.
The positive multiples of 10 that are no larger than $10^2 = 100$ are

$$10, 20, 30, 40, 50, 60, 70, 80, 90, 100$$

Of these, 10, 20, 30, and 40 are in earlier rows, so the smallest integer in Row 10 is 50.

(b) Consider an integer $n \geq 3$. Notice that n, $n - 1$, and $n - 2$ are all positive integers.

Factoring, the quantity $n^2 - n$, we get $n^2 - n = n(n - 1)$.
Since $n > n - 1$, $n(n - 1) > (n - 1)(n - 1) = (n - 1)^2$, which means that $n(n - 1)$ is too large to be in Row $n - 1$.
This also implies that $n(n - 1)$ is larger than each of 1^2, 2^2, 3^2, and so on up to $(n - 2)^2$, so we conclude that $n(n - 1)$ is not in any of the first $n - 1$ rows.
We also have that $n(n - 1)$ is a multiple of n, and since n is positive, $n^2 - n < n^2$.
Therefore, $n^2 - n$ is a multiple of n so it satisfies (i), it does not exceed n^2 so it satisfies (ii), and it is not in an earlier row, so it satisfies (iii).
This shows that $n^2 - n$ is in Row n.

By factoring $n^2 - 2n$, we have $n^2 - 2n = n(n - 2)$.
Similar to the reasoning above, $n > n - 2$ implies that $n(n - 2) > (n - 2)(n - 2) = (n - 2)^2$.
This implies that $n^2 - 2n$ is larger than k^2 for all positive integers k with $k \leq n - 2$.
As well, since n is positive, $n^2 - 2n < n^2$, so $n^2 - 2n$ is a multiple of n that does not exceed n^2.
By the given conditions on Row n, $n^2 - 2n$ must be in Row n provided it is not in an earlier row. We have already argued that it cannot be in any of Rows 1 through $n - 2$.
Therefore, we can show that $n^2 - 2n$ is in Row n by showing that it is *not* in Row $n - 1$.

We will now justify that for all $n \geq 3$, the integer $n^2 - 2n$ is not a multiple of $n - 1$, which implies $n^2 - 2n$ is not in Row $n - 1$.
Consider the quantity $(n - 1)^2$. We can expand it as follows:

$$\begin{aligned}
(n - 1)^2 &= (n - 1)(n - 1) \\
&= (n - 1)n + (n - 1)(-1) \\
&= n^2 - n + (-n) + (-1)(-1) \\
&= n^2 - 2n + 1
\end{aligned}$$

We can now rearrange $(n - 1)^2 = n^2 - 2n + 1$ to get $n^2 - 2n = (n - 1)^2 - 1$, which shows that $n^2 - 2n$ and $(n - 1)^2$ are consecutive integers.
Note that any two multiples of $n - 1$ must differ by at least $n - 1$.
We are assuming that $n \geq 3$, which implies that $n - 1 \geq 2$, and so any two multiples of $n - 1$ must differ by at least 2.
Since $(n - 1)^2$ and $n^2 - 2n$ differ by 1, they cannot both be multiples of $n - 1$.
The integer $(n - 1)^2$ is a multiple of $n - 1$, and so we conclude that $n^2 - 2n$ is not a multiple of $n - 1$.

(c) We will show that $n = 30$ is the largest integer with the property that $n^2 - 10n$ is not in Row n.
When $n = 30$, $n^2 - 10n = 30^2 - 10 \times 30 = 600$.
Since $600 = 24 \times 25$, we see that 600 is a multiple of 25 that is less than 25^2.
Therefore, 600 is in Row 25 or an earlier row, so $n = 30$ has the property that $n^2 - 10n$ is not in Row n.

In order to show that $n = 30$ is the largest such integer, we need to show that all integers n with the property that $n^2 - 10n$ is not in Row n satisfy $n \leq 30$.

For the remainder of the solution, we will assume that n is a positive integer with the property that $n^2 - 10n$ is not in Row n. We will deduce from this assumption that $n \leq 30$.
Note that if $n \leq 10$, then $n^2 - 10n = n(n - 10) \leq 0$, which means $n^2 - 10n$ is not in *any*

row, so we will also assume that $n \geq 11$ in order to ensure that $n^2 - 10n > 0$.

Since $n(n - 10) = n^2 - 10n$ and n is positive, we must have $n^2 - 10n < n^2$. This means $n^2 - 10n$ must occur somewhere in the first n rows.

Since $n > n-10$ and $n-10$ is positive, we have that $n(n-10) > (n-10)(n-10) = (n-10)^2$. This shows that $n(n - 10)$ is too large to be in Row m for any $m \leq n - 10$.

The previous two sentences combine to show that $n^2 - 10n$ must be in one of Row $n - 9$, Row $n - 8$, and so on up to Row n. We are assuming that $n^2 - 10n$ is not in Row n, so it must be in one of Rows $n - 9$ through $n - 1$.

This means we have 9 cases to consider: $n^2 - 10n$ is in Row $n - 9$, in Row $n - 8$, and so on to Row $n - 1$.

For each integer k from 1 through 9, we will consider the case when $n^2 - 10n$ is in Row $n-k$, and in each of these cases, we will show that $n \leq 30$. This will show that no matter which of the 9 rows $n^2 - 10n$ is in, it cannot be larger than 30.

Suppose $n^2 - 10n$ is in Row $n - 9$. Among other things, this means $n^2 - 10n$ must be a multiple of $n - 9$.

The expression $n^2 - 10n$ does not have an obvious factor of $n - 9$. It will be useful to find an algebraic expression that is a multiple of $n - 9$ and is "close" to $n^2 - 10n$.

After some trial and error, you might notice that $(n - 9)(n - 1) = n^2 - 10n + 9$, which can be rearranged to get $9 = (n - 9)(n - 1) - (n^2 - 10n)$.

We are assuming n^2-10n is a multiple of $n-9$, so the expression $(n-9)(n-1)-(n^2-10n)$ is the difference of two multiples of $n - 9$.

The difference between two multiples of $n - 9$ is itself a multiple of 9, which means $9 = (n - 9)(n - 1) - (n^2 - 10n)$ must be a multiple of $n - 9$.

If 9 is a multiple of $n - 9$, then we must have that $9 \geq n - 9$, which can be rearranged to get that $n \leq 18$.

We have now shown that if $n^2 - 10n$ is in Row $n - 9$, then $n \leq 18$, and so $n \leq 30$.

This kind of reasoning can be applied to the remaining eight cases.

For example, suppose $n^2 - 10n$ is in Row $n - 8$.

Observe that $(n - 8)(n - 2) = n^2 - 10n + 16$, and so $16 = (n - 8)(n - 2) - (n^2 - 10n)$.

If $n^2 - 10n$ in Row $n - 8$, then $n^2 - 10n$ is a multiple of $n - 8$, which means the expression $16 = (n - 8)(n - 2) - (n^2 - 10n)$ is a multiple of $n - 8$.

If 16 is a multiple of $n - 8$, then $n - 8 \leq 16$, so $n \leq 24$ which implies $n \leq 30$.

We will now quickly include the results of examining the remaining seven cases. The calculations are similar to those for Rows $n - 9$ and $n - 8$ and are not included.

$n^2 - 10n$ is in Row $n - 7$: $21 = (n - 7)(n - 3) - (n^2 - 10n)$, so 21 is a multiple of $n - 7$. Therefore, $n - 7 \leq 21$ or $n \leq 28$, and so $n \leq 30$.

$n^2 - 10n$ is in Row $n - 6$: $24 = (n - 6)(n - 4) - (n^2 - 10n)$, so 24 is a multiple of $n - 6$. Therefore, $n - 6 \leq 24$ or $n \leq 30$.

$n^2 - 10n$ is in Row $n - 5$: $25 = (n - 5)(n - 5) - (n^2 - 10n)$, so 25 is a multiple of $n - 5$. Therefore, $n - 5 \leq 25$ or $n \leq 30$.

$n^2 - 10n$ is in Row $n - 4$: $24 = (n - 4)(n - 6) - (n^2 - 10n)$, so 24 is a multiple of $n - 4$. Therefore, $n - 4 \leq 24$ or $n \leq 28$, and so $n \leq 30$.

$n^2 - 10n$ is in Row $n - 3$: $21 = (n - 3)(n - 7) - (n^2 - 10n)$, so 21 is a multiple of $n - 3$. Therefore, $n - 3 \leq 21$, or $n \leq 24$, and so $n \leq 30$.

$n^2 - 10n$ is in Row $n - 2$: $16 = (n - 2)(n - 8) - (n^2 - 10n)$, so 16 is a multiple of $n - 2$.

Therefore, $n - 2 \le 16$ or $n \le 18$, and so $n \le 30$.

$\underline{n^2 - 10n \text{ is in Row } n - 1}$: $9 = (n-1)(n-9) - (n^2 - 10n)$, so 9 is a multiple of $n - 1$. Therefore, $n - 1 \le 9$ or $n \le 10$, and so $n \le 30$.

To recap, we assumed that $n^2 - 10n$ is not in Row n, then deduced that it must be in Row k for some k from $n - 9$ through $n - 1$. We then considered each each of these 9 possibilities individually and showed that $n \le 30$ in each case.

Therefore, if $n^2 - 10n$ is not in Row n, then $n \le 30$. Since $n = 30$ has the property that $n^2 - 10n$ is not in Row n, we can conclude that 30 is the largest integer with this property.