Problem of the Month

Problem 2: November 2022

Let $\phi = \dfrac{1+\sqrt{5}}{2} \approx 1.61803$. For integers $d_k, d_{k-1}, \ldots, d_1, d_0, d_{-1}, \ldots, d_{-r}$, each equal to 0 or 1, the expression

$$(d_k d_{k-1} \cdots d_2 d_1 d_0 . d_{-1} d_{-2} \cdots d_{-r})_\phi$$

is called a *base ϕ expansion* and represents the real number

$$d_k\phi^k + d_{k-1}\phi^{k-1} + \cdots + d_1\phi + d_0 + d_{-1}\phi^{-1} + d_{-2}\phi^{-2} + \cdots + d_{-r}\phi^{-r}$$

The integers d_k through d_{-r} are called the *digits* of the expansion. For example, the base ϕ expansion 1101.011_ϕ represents the real number

$$(1 \times \phi^3) + (1 \times \phi^2) + (0 \times \phi) + 1 + (0 \times \phi^{-1}) + (1 \times \phi^{-2}) + (1 \times \phi^{-3})$$

which can be simplified to get

$$\begin{aligned}
\phi^3 + \phi^2 + 1 + \frac{1}{\phi^2} + \frac{1}{\phi^3} &= \left(\frac{1+\sqrt{5}}{2}\right)^3 + \left(\frac{1+\sqrt{5}}{2}\right)^2 + 1 + \left(\frac{2}{1+\sqrt{5}}\right)^2 + \left(\frac{2}{1+\sqrt{5}}\right)^3 \\
&= \frac{16+8\sqrt{5}}{8} + \frac{6+2\sqrt{5}}{4} + 1 + \frac{4}{6+2\sqrt{5}} + \frac{8}{16+8\sqrt{5}} \\
&= (2+\sqrt{5}) + \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) + 1 + \left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) - (2-\sqrt{5}) \\
&= 4 + 2\sqrt{5}
\end{aligned}$$

and so $1101.011_\phi = 4 + 2\sqrt{5}$.

(a) What are the real numbers represented by 1011_ϕ and 10000_ϕ?

(b) Find a base ϕ expansion of the real number $4 + 3\sqrt{5}$.

(c) Show that $\phi^2 = \phi + 1$ and use this to deduce that $\phi^{n+1} = \phi^n + \phi^{n-1}$ for all integers n.

(d) Show that every positive integer has a base ϕ expansion and find a base ϕ expansion for each positive integer from 1 through 10. One approach is to prove and use the following two facts.

- If a real number n has a base ϕ expansion, then it has a base ϕ expansion that does not have two consecutive digits equal to 1.
- If a real number n has a base ϕ expansion, then it has a base ϕ expansion that has its units digit, d_0, equal to 0.