Problem of the Month

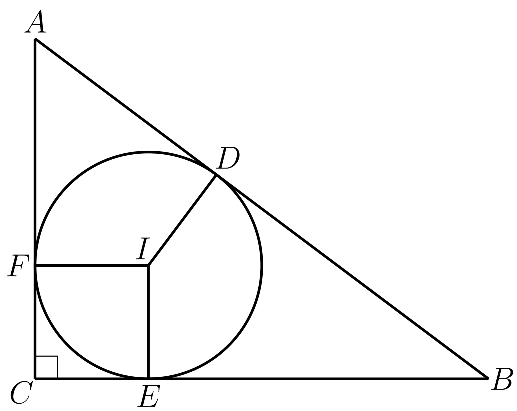
Solution to Problem 1: October 2022

Note: In this solution, we will denote by $|\triangle ABC|$ the area of $\triangle ABC$. We will also use the following facts about circles.

Fact 1: Suppose a circle has centre O and P is a point on its circumference. The tangent to the circle at point P is perpendicular to the radius from O to P.

Fact 2: For any point Q outside of a circle, there are exactly two tangents to the circle that pass through Q. If the points of tangency are A and B, then $AQ = BQ$.

(a) Below is a picture of the triangle. Its vertices are labelled by A, B, and C with the right angle at C, $AC = 3$, $BC = 4$, and $AB = 5$. The centre of the incircle is labelled by I, and AB, BC, and AC are tangent to the incircle at D, E, and F, respectively.



Since the perimeter is $3 + 4 + 5 = 12$, the semiperimeter is $s = \dfrac{12}{2} = 6$.

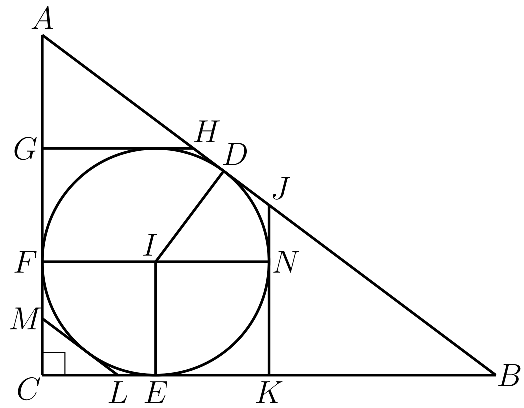
To compute r, we first use Fact 1 to get that $\angle IFC = \angle IEC = 90^\circ$. We are assuming that $\angle FCE = 90^\circ$ as well, and since the sum of the angles of a quadrilateral is always 360°, $\angle FIE = 90^\circ$. Therefore, $CFIE$ is a rectangle. Since IF and IE are radii of the incircle, $IF = IE = r$. Since $CFIE$ is a rectangle, $CF = CE = r$.

Using that $CF = CE = r$, we get $BE = BC - CE = 4 - r$ and $AF = AC - CF = 3 - r$. By Fact 2, $BD = BE$ and $AD = AF$, and since $AB = 5$,

$$\begin{aligned} 5 &= AB \\ &= AD + BD \\ &= AF + BE \\ &= (3 - r) + (4 - r) \\ &= 7 - 2r \end{aligned}$$

This gives $5 = 7 - 2r$, which can be solved for r to get $r = 1$.

In the diagram below, the Seraj hexagon has been added. The side of the Seraj hexagon that is parallel to AB (and tangent to the incircle) intersects BC at L and AC at M. The side of the Seraj hexagon that is parallel to BC intersects AC at G and AB at H. The side of the Seraj hexagon that is parallel to AC intersects AB at J and BC at K. The point of tangency of JK with the circle is labelled by N.



To compute the area of the Seraj hexagon, we will compute the areas of $\triangle MLC$, $\triangle AHG$, and $\triangle JBK$ and subtract their combined area from the area of $\triangle ABC$.

To compute the area of $\triangle MLC$, we first set $CM = x$ and $CL = y$. Since LM is parallel to AB, we have that $\angle CML = \angle CAB$ and $\angle CLM = \angle CBA$. Since the two share a right angle at C, $\triangle MLC$ is similar to $\triangle ABC$. Therefore, $\dfrac{y}{x} = \dfrac{CL}{CM} = \dfrac{BC}{AC} = \dfrac{4}{3}$, or $y = \dfrac{4}{3}x$.

From earlier, we have that $CF = CE = 1$, which means $FM = 1 - x$ and $EL = 1 - y$. By Fact 2, $LM = EL + FM = 2 - x - y$. By the Pythagorean theorem applied to $\triangle CML$ and using that $y = \dfrac{4}{3}x$, we have

$$\begin{aligned}
CM^2 + CL^2 &= LM^2 \\
x^2 + y^2 &= (2 - x - y)^2 \\
x^2 + \left(\frac{4}{3}x\right)^2 &= \left(2 - \frac{7}{3}x\right)^2 \\
x^2 + \frac{16}{9}x^2 &= 4 - \frac{28}{3}x + \frac{49}{9}x^2 \\
0 &= \frac{8}{3}x^2 - \frac{28}{3}x + 4 \\
0 &= 2x^2 - 7x + 3 && \text{(multiply by } \tfrac{3}{4}\text{)} \\
0 &= (x - 3)(2x - 1)
\end{aligned}$$

If $x - 3 = 0$, then M is at A, but AB and LM are distinct parallel lines, so they have no points in common. Therefore, $2x - 1 = 0$ or $x = \dfrac{1}{2}$, so $y = \dfrac{4}{3} \times \dfrac{1}{2} = \dfrac{2}{3}$. We can now compute

$$|\triangle CML| = \frac{1}{2}xy = \frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{6}$$

We now compute the area of $\triangle JBK$. By Fact 1, $\angle IEK = \angle INK = 90^\circ$, and since JK is parallel to AC, $\angle EKN = 90^\circ$ as well. By reasoning similar to earlier, we conclude that $INKE$ is a square of side-length r, which means $EK = r$. We also have that $CE = r$, so this means $BK = BC - CE - EK = 4 - 1 - 1 = 2$.

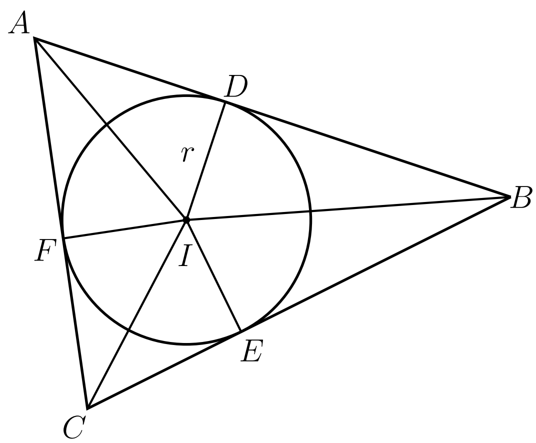
Since JK is parallel to AC, $\triangle JBK$ is similar to $\triangle ABC$ by reasoning similar to that which was used to show that $\triangle MLC$ was similar to $\triangle ABC$. This means $\frac{JK}{AC} = \frac{BK}{BC}$ or $JK = \frac{AC \times BK}{BC}$. Substituting $AC = 3$, $BK = 2$, and $BC = 4$ into this equation, we get $JK = \frac{3}{2}$. Therefore,

$$|\triangle JBK| = \frac{1}{2} \times BK \times JK = \frac{1}{2} \times 2 \times \frac{3}{2} = \frac{3}{2}$$

Using very similar reasoning, one can show that $|\triangle AHG| = \frac{2}{3}$. Therefore, the area of the Seraj hexagon is

$$|\triangle ABC| - |\triangle MLC| - |\triangle JBK| - |\triangle AHG| = \frac{1}{2} \times 4 \times 3 - \frac{1}{6} - \frac{3}{2} - \frac{2}{3} = \frac{11}{3}$$

(b) In the diagram below, a triangle has its incircle drawn. Radii are drawn from the centre of the incircle, I, to the points of tangency of AB, BC, and AC, which are labelled D, E, and F, respectively. As well, I is connected by line segments to each vertex of the triangle.

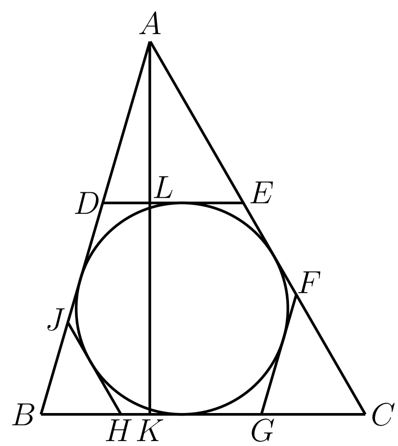


Set $AB = c$, $BC = a$, and $AC = b$. With this notation, the semiperiemter of $\triangle ABC$ is $s = \frac{a+b+c}{2}$. Since ID, IE, and IF are radii to points of tangency, they are perpendicular to AB, BC, and AC, respectively. Therefore, they are altitudes of $\triangle ABI$, $\triangle BCI$, and $\triangle CAI$, respectively. These three triangles compose the entirety of $\triangle ABC$ with no overlap, so the area of $\triangle ABC$ is equal to the sum of their areas. Therefore,

$$\begin{aligned} |\triangle ABC| &= \frac{1}{2} \times AB \times ID + \frac{1}{2} \times BC \times IE + \frac{1}{2} \times AC \times IF \\ &= \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br \\ &= r \times \frac{a+b+c}{2} \\ &= rs \end{aligned}$$

As an example demonstrating this formula, consider $\triangle ABC$ from part (a). Using the usual area formula, $|\triangle ABC| = \frac{1}{2} \times 3 \times 4 = 6$. From part (a), its semiperimeter is $\frac{3+4+5}{2} = 6$ and inradius is $r = 1$, so $rs = 1 \times 6 = 6$, which is the correct area.

(c) In the diagram below, $\triangle ABC$ has its incircle and Seraj hexagon drawn. The side of the Seraj hexagon that is parallel to BC intersects AB at D and AC at E. The side parallel to AB intersects AC at F and BC at G. The side parallel to AC intersects BC at H and AB at J. The altitude of $\triangle ABC$ from A is also drawn and its points of intersection with BC and DE are labelled by K and L, respectively[1].



By construction, DE is parallel to BC. By reasoning similar to that which was used in earlier parts, this means $\triangle ABC$ is similar to $\triangle ADE$ and $\triangle ABK$ is similar to $\triangle ADL$. These two pairs of similar triangles imply that $\frac{DE}{BC} = \frac{AD}{AB}$ and $\frac{AD}{AB} = \frac{AL}{AK}$. We will call this common ratio k. The area of $\triangle ADE$ can be computed in terms of k and the area of $\triangle ABC$ as follows

$$\begin{aligned}
|\triangle ADE| &= \frac{1}{2}(DE)(AL) \\
&= \frac{1}{2}\left(k(BC)\right)\left(k(AK)\right) \\
&= k^2\left(\frac{1}{2} \times BC \times AK\right) \\
&= k^2|\triangle ABC|
\end{aligned}$$

We next examine the quantity k. Since DE and BC are different parallel tangents to the incircle and KL is perpendicular to both lines, the length of KL must be equal to the diameter of the incircle (you may want to think about why this is true). The diameter of the incircle is $2r$, so $KL = 2r$, which means $AL = AK - 2r$.

Therefore, we have $k = \frac{AL}{AK} = \frac{AK - 2r}{AK} = 1 - \frac{2r}{AK}$. From part (b), $|\triangle ABC| = rs$, but from the usual formula for the area of a triangle we also have $|\triangle ABC| = \frac{1}{2}(BC)(AK)$. This implies $2rs = (BC)(AK)$ or $AK = \frac{2rs}{BC}$. Substituting into the formula for k above, we have that

$$k = 1 - \frac{2r}{AK} = 1 - \frac{2r(BC)}{2rs} = 1 - \frac{BC}{s}$$

[1]If $\angle ABC$ or $\angle ACB$ is obtuse, then AK intersects some extensions DE and BC and not the line segments themselves. We leave it to the reader to verify that the argument that follows works even if AK does not pass through DE and BC.

If we set $AB = c$, $BC = a$, and $AC = b$, then we get $k = 1 - \dfrac{a}{s}$. Earlier, we showed that $|\triangle ADE| = k^2|\triangle ABC|$, so

$$|\triangle ADE| = \left(1 - \frac{a}{s}\right)^2 |\triangle ABC|$$

By similar reasoning, we also have

$$|\triangle JBH| = \left(1 - \frac{b}{s}\right)^2 |\triangle ABC|$$
$$|\triangle FGC| = \left(1 - \frac{c}{s}\right)^2 |\triangle ABC|$$

The area of the Seraj hexagon is equal to the area of $\triangle ABC$ minus the combined area of these three triangles, so using the formulas just above as well as $|\triangle ABC| = rs$, we have

$$\begin{aligned} |DEFGHJ| &= |\triangle ABC| - |\triangle ADE| - |\triangle JBH| - |\triangle FGC| \\ &= |\triangle ABC| \left[1 - \left(1 - \frac{a}{s}\right)^2 - \left(1 - \frac{b}{s}\right)^2 - \left(1 - \frac{c}{s}\right)^2\right] \\ &= rs \left[1 - \left(1 - \frac{a}{s}\right)^2 - \left(1 - \frac{b}{s}\right)^2 - \left(1 - \frac{c}{s}\right)^2\right] \end{aligned}$$

(d) As suggested in the hint, we will show that $3(x^2 + y^2 + z^2) \geq (x + y + z)^2$ for all real numbers x, y, and z. To see this, note that the inequality is equivalent to the inequality

$$3(x^2 + y^2 + z^2) - (x + y + z)^2 \geq 0$$

which, after expanding and rearranging the left side, is the same as

$$2(x^2 + y^2 + z^2) - 2(xy + yz + zx) \geq 0$$

After further manipulation, this is equivalent to

$$(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$$

Therefore, the given inequality is true exactly when $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$. The sum of three squares is always non-negative, so this inequality is always true. Therefore, the given inequality is true for all real numbers x, y, and z, as claimed. Moreover, the only way that the quantity $(x - y)^2 + (y - z)^2 + (z - x)^2$ can be equal to 0 is when $x = y = z$, and hence, the only way that $3(x^2 + y^2 + z^2) = (x + y + z)^2$ is when $x = y = z$.

Dividing the expression for the area of the Seraj hexagon from part (c) by rs, we get that the ratio of the area of the Seraj hexagon to that of the triangle is

$$1 - \left(1 - \frac{a}{s}\right)^2 - \left(1 - \frac{b}{s}\right)^2 - \left(1 - \frac{c}{s}\right)^2$$

Using the inequality established above, we can multiply by $-\dfrac{1}{3}$ to get that for any real numbers x, y, and z, $-x^2 - y^2 - z^2 \leq -\dfrac{1}{3}(x + y + z)^2$. Applying this with $x = 1 - \dfrac{a}{s}$,

$y = 1 - \frac{b}{s}$, and $z = 1 - \frac{c}{s}$, we get that

$$\begin{aligned}
1 - \left(1 - \frac{a}{s}\right)^2 - \left(1 - \frac{b}{s}\right)^2 - \left(1 - \frac{c}{s}\right)^2 &= 1 - x^2 - y^2 - z^2 \\
&\leq 1 - \frac{1}{3}(x + y + z)^2 \\
&= 1 - \frac{1}{3}\left(1 - \frac{a}{s} + 1 - \frac{b}{s} + 1 - \frac{c}{s}\right)^2 \\
&= 1 - \frac{1}{3}\left(3 - \frac{a + b + c}{s}\right)^2 \\
&= 1 - \frac{1}{3}\left(3 - \frac{2s}{s}\right)^2 \\
&= 1 - \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}$$

Therefore, the ratio is at most $\frac{2}{3}$ in every triangle. By the remark at the end of the proof of the inequality, the ratio equals $\frac{2}{3}$ exactly when $1 - \frac{a}{s} = 1 - \frac{b}{s} = 1 - \frac{c}{s}$. Since s is always nonzero, this is equivalent to $a = b = c$. Therefore, the ratio is maximized when the triangle is equilateral. It is not difficult to explicitly show that the area of the Seraj hexagon in an equilateral triangle is equal to two thirds of the area of the triangle.