The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Hypatia Contest

(Grade 11)

Tuesday, April 12, 2022
(in North America and South America)

Wednesday, April 13, 2022
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes

Do not open this booklet until instructed to do so.

Number of questions: 4 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 
   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by 
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions must be supplied by your supervising teacher and inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked  , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked  , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

Useful Fact:
It may be helpful to know that $2^n \geq n + 1$ for all positive integers n.

1. A *regular hexagon* is a polygon that has six sides with equal length and six interior angles with equal measure. In Figure 1, regular hexagon $ABCDEF$ has side length $2x$ and its vertices lie on the circle with centre O. The diagonals AD, BE and CF divide $ABCDEF$ into six congruent equilateral triangles.

 (a) In terms of x, what is the radius of the circle?

 (b) The midpoint of side AB is labelled M, as shown in Figure 2. In terms of x, what is the length of OM?

 (c) In terms of x, what is the area of hexagon $ABCDEF$?

 (d) The region that lies inside the circle and outside hexagon $ABCDEF$ is shaded, as shown in Figure 3. The area of this shaded region is 123. Rounded to the nearest tenth, determine the value of x.

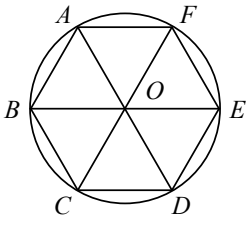

Figure 1

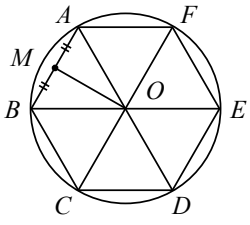

Figure 2

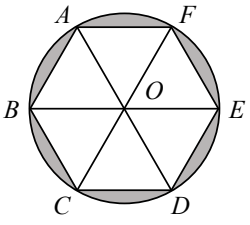

Figure 3

2. With 1 kg of muffin batter, exactly 24 mini muffins and 2 large muffins can be made. With 2 kg of muffin batter, exactly 36 mini muffins and 6 large muffins can be made.


(a) With 2 kg of muffin batter, exactly 48 mini muffins and n large muffins can also be made. What is the value of n?


(b) With x kg of muffin batter, exactly 84 mini muffins and 10 large muffins can be made. What is the value of x?


(c) Determine how many mini muffins can be made using the same amount of batter that is needed to make 7 large muffins.

3. A sequence is created in such a way that

- a real number is chosen as the first number in the sequence, and
- each of the following numbers in the sequence is generated by applying a function to the previous number in the sequence.

For example, if the first number in a sequence is 1 and the following numbers are generated by the function $x^2 - 5$, then the first three numbers in the sequence are $1, -4$ and 11 since $1^2 - 5 = -4$ and $(-4)^2 - 5 = 11$.


(a) The first number in a sequence is 3 and the sequence is generated by the function $x^2 - 3x + 1$. What are the first four numbers in the sequence?


(b) The number 7 is the third number in a sequence generated by the function $x^2 - 4x + 7$. What are all possible first numbers in the sequence?


(c) The first number in a sequence is c and the sequence is generated by the function $x^2 - 7x - 48$. If all numbers in the sequence are equal to c, determine all possible values of c.


(d) A sequence generated by the function $x^2 - 12x + 39$ alternates between two different numbers. That is, the sequence is $a, b, a, b, a, b, \ldots$, with $a \neq b$. Determine all possible values of a.

4. Every integer $N > 1$ can be written as $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$, where k is a positive integer, $p_1 < p_2 < p_3 < \cdots < p_k$ are prime numbers, and $a_1, a_2, a_3, \ldots, a_k$ are positive integers. For example, $1400 = 2^3 5^2 7^1$.

The number of positive divisors of N is denoted by $f(N)$. It is known that

$$f(N) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_k)$$


(a) How many positive divisors does 240 have? That is, what is the value of $f(240)$?


(b) Define an integer $N > 1$ to be *refactorable* if it is divisible by $f(N)$. For example, both 6 and 8 have 4 positive divisors, so 8 is refactorable and 6 is not refactorable. This is because 8 is divisible by 4, but 6 is not divisible by 4. Determine all refactorable numbers N with $f(N) = 6$.


(c) Determine the smallest refactorable number N with $f(N) = 256$.


(d) Show that for every integer $m > 1$, there exists a refactorable number N such that $f(N) = m$.

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For students...

Thank you for writing the 2022 Hypatia Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2022.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2022/2023 contests
- Register your students for the Canadian Senior and Intermediate Mathematics Contests which will be written in November
- Look at our free online courseware for senior high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results