The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2022 Fryer Contest

Tuesday, April 12, 2022
(in North America and South America)

Wednesday, April 13, 2022
(outside of North America and South America)

Solutions

1. (a) Each hit adds 7 points to Shane's score, and so 4 hits adds $7 \times 4 = 28$ points to his score.
For each miss, 3 points are subtracted from Shane's score, and so 2 misses subtracts $3 \times 2 = 6$ points from his score.
A player's score begins at 0, and so after these 6 throws, Shane's score is $28 - 6 = 22$.

(b) *Solution 1*
After exactly h hits and 6 misses, Susan's score is given by the expression $7 \times h - 3 \times 6$ or $7h - 18$.
After these throws, Susan's score is 59, and so $7h - 18 = 59$.
Solving this equation, we get $7h = 59 + 18$ or $7h = 77$, and so $h = 11$.

Solution 2
Susan's 6 misses decrease her score by $3 \times 6 = 18$ points.
Since her score is 59, then Susan must have scored $59 + 18 = 77$ points from hits.
Since each hit is worth 7 points, then the value of h is $77 \div 7 = 11$.

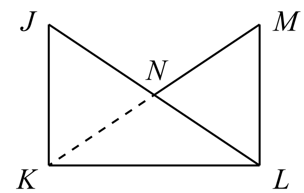
(c) *Solution 1*
We begin by making an initial guess at the value of m, and then systematically adjust the value upward or downward as needed.
If we begin with $m = 3$ for example, then the number of hits is $20 - 3 = 17$ and Souresh's score is $7 \times 17 - 3 \times 3 = 119 - 9 = 110$.
Since this score is greater than 105, then the value of m is greater than 3 (more misses means fewer hits and a lower score).
When $m = 4$, the number of hits is $20 - 4 = 16$ and Souresh's score is $7 \times 16 - 3 \times 4 = 112 - 12 = 100$.
Since this score is greater than 85 and less than 105, then 4 is a possible value of m.
When $m = 5$, the number of hits is $20 - 5 = 15$ and Souresh's score is $7 \times 15 - 3 \times 5 = 105 - 15 = 90$.
This score is also greater than 85 and less than 105, and so 5 is a possible value of m.
When $m = 6$, the number of hits is 14 and Souresh's score is $7 \times 14 - 3 \times 6 = 98 - 18 = 80$.
This score is less than 85 and so 6 is not a possible value of m.
Continuing to increase the number of misses will further decrease Souresh's score, and thus the only possible values are $m = 4$ and $m = 5$.

Solution 2
Since Souresh makes 20 throws and m of these throws are misses, then the remaining $20 - m$ throws are hits.
After exactly $20 - m$ hits and m misses, Souresh's score is given by the expression $7(20 - m) - 3m$ or $140 - 10m$.
Since Souresh's score $(140 - 10m)$ is greater than 85, then $10m$ must be less than 55 (note that $140 - 85 = 55$) and so $m < \frac{55}{10}$.
Since $\frac{55}{10} = 5\frac{1}{2}$ and m is a positive integer, then $m \leq 5$.
Since Souresh's score is less than 105, then $10m$ must be greater than 35 (note that $140 - 105 = 35$) and so $m > \frac{35}{10}$.
Since $\frac{35}{10} = 3\frac{1}{2}$ and m is a positive integer, then $m \geq 4$.
Thus, m is a positive integer and $4 \leq m \leq 5$, and so $m = 4$ or $m = 5$.
(We can check that when $m = 4$, Souresh's score is $7 \times 16 - 3 \times 4 = 112 - 12 = 100$, and when $m = 5$, Souresh's score is $7 \times 15 - 3 \times 5 = 105 - 15 = 90$.)

2. (a) *Solution 1*
The area of $ABGH$ is equal to the sum of the areas of $ABCD$ and $EFGH$ minus the area of overlap $EFCD$ since it is counted twice in this sum.
Thus, the area of $ABGH$ is $(13 \text{ cm}^2) + (13 \text{ cm}^2) - (5 \text{ cm}^2) = 21 \text{ cm}^2$.

Solution 2
The area of $ABCD$ is 13 cm^2 and is equal to the sum of the areas of $ABFE$ and $EFCD$.
Since the area of $EFCD$ is 5 cm^2, then the area of $ABFE$ is $(13 \text{ cm}^2) - (5 \text{ cm}^2) = 8 \text{ cm}^2$.
The area of $ABGH$ is equal to the sum of the areas of $ABFE$ and $EFGH$ or $(8 \text{ cm}^2) + (13 \text{ cm}^2) = 21 \text{ cm}^2$.

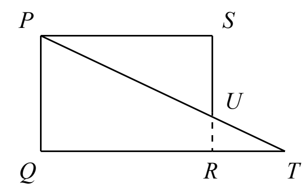
(b) Let the area of the overlapped region, $\triangle KLN$, be $x \text{ cm}^2$.
The area of $\triangle KLN$ is equal to half of the area of $\triangle JKL$, and so the area of $\triangle JKN$ is also $x \text{ cm}^2$.
Since $\triangle JKL$ and $\triangle MLK$ are identical, then the area of $\triangle MLN$ is also $x \text{ cm}^2$.
Thus, the area of the figure $JKLMN$ is $3x \text{ cm}^2$, and so $3x = 48$ or $x = 16$.



Since $\triangle JKL$ is right-angled at K, then its area is given by $\frac{1}{2}(JK)(KL)$.
Since the area of $\triangle JKL$ is $2x \text{ cm}^2 = 32 \text{ cm}^2$, then $\frac{1}{2}(JK)(KL) = 32 \text{ cm}^2$ or $\frac{1}{2}(6 \text{ cm})(KL) = 32 \text{ cm}^2$, and so $KL = \frac{32}{3}$ cm.

(c) We use the notation $|\triangle URT|$ to denote the area of $\triangle URT$, and similar notation for other areas.
Since $|PQRS| + |\triangle URT| = |PQTUS|$, then

$$|\triangle URT| = |PQTUS| - |PQRS| = 117 \text{ cm}^2 - 108 \text{ cm}^2 = 9 \text{ cm}^2.$$



Since $|PQRU| + |\triangle URT| = |\triangle PQT|$, then

$$|PQRU| = |\triangle PQT| - |\triangle URT| = 81 \text{ cm}^2 - 9 \text{ cm}^2 = 72 \text{ cm}^2.$$

3. (a) The prime factorization of $675 = 3^3 \times 5^2$, and so 675 has $4 \times 3 = 12$ positive factors.

(b) The positive integer n has a total of $4 + 14 = 18$ positive factors.
Since n has the positive factors $9 = 3^2$, 11, $15 = 3 \times 5$, and $25 = 5^2$, then the prime factorization of n must include at least 2 factors of 3, at least 2 factors of 5, and at least 1 factor of 11.
In other words, n must be divisible by $3^2 \times 5^2 \times 11$.
Suppose $n = 3^2 \times 5^2 \times 11$. Then n has $3 \times 3 \times 2 = 18$ positive factors, as required.
If n contained additional factors, then it would have more than 18 positive factors.
Thus, $n = 3^2 \times 5^2 \times 11 = 2475$.

(c) Suppose that m is a positive integer less than 500 that has exactly $2 + 10 = 12$ positive factors.
Since m has the positive factors 2 and $9 = 3^2$, then the prime factorization of m must include at least 1 factor of 2, and at least 2 factors of 3.
In other words, m must be divisible by 2×3^2.
To begin, suppose that m has exactly 2 distinct prime factors.
That is, suppose that $m = 2^a \times 3^b$ where a and b are integers with $a \geq 1$ and $b \geq 2$.
In this case, m has $(a + 1)(b + 1) = 12$ positive factors.
Since $a \geq 1$ and $b \geq 2$, then $a + 1 \geq 2$ and $b + 1 \geq 3$.

Using these restrictions, there are exactly three possibilities for which $(a+1)(b+1)=12$. These are

$$a+1=2 \text{ and } b+1=6, \text{ which gives } a=1 \text{ and } b=5$$
$$a+1=3 \text{ and } b+1=4, \text{ which gives } a=2 \text{ and } b=3$$
$$a+1=4 \text{ and } b+1=3, \text{ which gives } a=3 \text{ and } b=2$$

If $a=1$ and $b=5$, then $m=2\times 3^5=486$.
If $a=2$ and $b=3$, then $m=2^2\times 3^3=108$.
If $a=3$ and $b=2$, then $m=2^3\times 3^2=72$.
Since each of these values is less than 500, then there are 3 positive integers that satisfy the given conditions, in this case.

Next, suppose that m has exactly 3 distinct prime factors.
That is, suppose that $m=2^a\times 3^b\times p^c$ where p is a prime number not equal to 2 or 3, and a, b and c are integers with $a\geq 1$, $b\geq 2$ and $c\geq 1$.
If $a=1$, $b=2$ and $c=1$ (the minimum values possible for a,b,c), then $m=2\times 3^2\times p$.
In this case, m has $2\times 3\times 2=12$ positive factors, as required.
Increasing a, b or c increases the number of positive factors, and thus $a=1$, $b=2$ and $c=1$ is the only possibility for which m has 12 positive factors and 3 distinct prime factors.
If $a=1$, $b=2$ and $c=1$, then $m=2\times 3^2\times p=18p$.
For which prime numbers $p>3$ is $18p$ less than 500?
Since $18p<500$, then $p<\frac{500}{18}$ and so $p\leq 27$.
The prime numbers in this range are 5,7,11,13,17,19, and 23, which give 7 positive integers that satisfy the given conditions, in this case.

Finally, suppose that m has exactly 4 distinct prime factors.
That is, suppose that $m=2^a\times 3^b\times p^c\times q^d$ where p and q are different prime numbers not equal to 2 or 3, and a, b, c, d are integers with $a\geq 1$, $b\geq 2$, $c\geq 1$, and $d\geq 1$.
If $a=1$, $b=2$, $c=1$, and $d=1$ (the minimum values possible for a,b,c,d), then m has $2\times 3\times 2\times 2=24$ positive factors, which is a contradiction.
Increasing a, b, c, or d or increasing the number of distinct prime factors, increases the number of positive factors, and thus there are no possibilities for which m has 12 positive factors and 4 or more distinct prime factors.

Thus, the number of positive integers less than 500 that have the factors 2 and 9 and exactly ten other positive factors is $3+7=10$.

4. (a) Since one of the jars contains 0 beans, then all beans must be removed from the jar that contains 40 beans.
On Franco's turns, the number of beans that he removes is 1, 3, 4, 1, 3, 4, $\ldots$, and so on.
After Franco's 5 turns, he has removed a total of $1+3+4+1+3=12$ beans.
On Sarah's turns, the number of beans that she removes is 2, 5, 2, 5, 2, 5, $\ldots$, and so on.
After Sarah's 5 turns, she has removed a total of $2+5+2+5+2=16$ beans.
After a total of 10 turns, the total number of beans removed by Franco and Sarah is less than 40, and so we know that each of the two players was able to remove the required number of beans on each of their 5 turns.
After a total of 10 turns, the total number of beans left in the two jars is $40-(12+16)=40-28=12$.

(b) Since one of the jars contains 0 beans, then all beans must be removed from the jar that contains 384 beans.

Franco repeatedly removes $1, 3$ and 4 beans, which is a cycle of length 3.
Sarah repeatedly removes 2 and 5 beans, which is a cycle of length 2.
Since the lowest common multiple of 3 and 2 is 6, then after each player has had 6 turns (12 turns in total), they will each be back at the start of their cycle.
After the first 12 turns (6 turns for each player), Franco removes

$$1+3+4+1+3+4=2(1+3+4)=16 \text{ beans},$$

and Sarah removes

$$2+5+2+5+2+5=3(2+5)=21 \text{ beans}.$$

On each of the following groups of 12 turns, Franco will again remove 16 beans and Sarah will again remove 21 beans.
From the start of the game, together they will remove $16+21=37$ beans every 12 turns.
After 10 such groups of turns (a total of $10\times 12=120$ turns), the total number of beans removed is $10\times 37=370$, and so $384-370=14$ beans are left in the jar.
After these 120 turns, Franco and Sarah are at the beginning of their sequences and it is Franco's turn.
On the 121^{st} turn, Franco removes 1 bean, and so $14-1=13$ beans remain.
On the 122^{nd} turn, Sarah removes 2 beans, and so $13-2=11$ beans remain.
On the 123^{rd} turn, Franco removes 3 beans, and so $11-3=8$ beans remain.
On the 124^{th} turn, Sarah removes 5 beans, and so $8-5=3$ beans remain.
The next turn is Franco's and he is not able to remove the required number of beans, 4, and thus Franco loses.
Therefore, Sarah wins after exactly $n=124$ turns.

(c) We use the notation Tn to represent turn n, and (x,y) to represent the distribution of x beans in one jar and y in the other jar.
The game begins with the distribution $(17,6)$, which is a total of $17+6=23$ beans.
After T4 (2 turns each), Franco has removed $1+3=4$ beans and Sarah has removed $2+5=7$ beans.
Thus after T4, there are a total of $23-(4+7)=12$ beans remaining in the jars.
With 12 beans remaining, at least one of the jars must contain 6 or more beans, since if both jars contained fewer than 6 beans, then the total number of beans would be at most $5+5=10$.
Since at least one of the jars contains 6 or more beans, then on T5, Franco is able to remove 4 beans leaving a total of $12-4=8$ beans.
With 8 beans remaining, at least one of the jars must contain 4 or more beans, and so on T6, Sarah is able to remove 2 beans, and on T7, Franco is able to remove 1 bean.
After T7 (4 turns for Franco and 3 for Sarah), the total number of beans remaining in the jars is $8-(2+1)=5$.
Sarah must remove 5 beans on T8 (Sarah's 4^{th} turn).
If the distribution of beans is $(5,0)$, then Sarah will remove the 5 beans and win on T8.
However, if the distribution is $(4,1)$ or $(3,2)$, then Sarah will not be able to remove 5 beans and so Franco will win.

We summarize the solution to this point:

- It is not possible for either player to win on T1 through T6
- After T7, the total number of beans remaining in the jars is 5
- On T8, it is Sarah's turn and she must remove 5 beans

- After T7, Franco wins if he leaves distributions of $(4, 1)$ or $(3, 2)$
- After T7, Franco will lose if he leaves a distribution of $(5, 0)$ (Sarah will win on T8)

Thus, Franco has a winning strategy if he can ensure that the final 5 beans are not all in one jar, otherwise Sarah has a winning strategy.

Franco has the winning strategy.
We summarize his strategy and then explain why this strategy guarantees that Franco will always win.
Franco's strategy:

- Remove 1 bean from the jar containing 17 beans on T1
- Remove 3 beans from the jar containing the fewest number of beans on T3
- Remove beans from the jar containing the greatest number of beans on T5 and T7

On T1, Franco removes 1 bean from the jar containing 17 beans, and so the distribution is $(16, 6)$.
On T2, Sarah removes 2 beans from one of the jars, and so the distribution is $(14, 6)$ or $(16, 4)$.
On T3, Franco removes 3 beans from the jar containing the fewest number of beans.
Thus after exactly 3 turns, the jars contain $(14, 3)$, or they contain $(16, 1)$.
Through T4 to T7 inclusive, $5 + 4 + 2 + 1 = 12$ more beans are removed from the jars.
Thus in each of the above two cases, the jar containing the greatest number of beans (14 and 16) can not be emptied.
Can Sarah empty a jar that contains 3 beans or that contains 1 bean without Franco removing any beans from these jars?
Since Sarah can remove 2 or 5 beans on each of her turns, she will be unable to remove exactly 3 beans and she will similarly be unable to remove exactly 1 bean.
This means that for each of the above two cases, Sarah will not be able to empty either of the jars.
Therefore, after exactly 7 turns the jars will contain $(2, 3)$, or $(4, 1)$, and thus Sarah will lose since she will not be able to remove 5 beans on T8.

(d) There are a total of $2023 + 2022 = 4045$ beans at the start of the game.
As was shown in part (b), 37 beans are removed in each successive group of 12 turns beginning from the start of the game.
After 109 such groups of turns (a total of $109 \times 12 = 1308$ turns), the total number of beans removed is $109 \times 37 = 4033$, and so $4045 - 4033 = 12$ beans are left in the jars.
After these 1308 turns, Franco and Sarah are at the beginning of their sequences and it is Franco's turn.
With 12 beans remaining, at least one of the jars must still contain 6 or more beans, and so Franco and Sarah were able to remove their required number of beans on each of the first 1308 turns.

From this point in the game, the winner is determined by how the 12 beans are distributed between the two jars.
For example, assume that the distribution of the final 12 beans is $(12, 0)$.
Franco removes 1, Sarah removes 2, Franco removes 3, Sarah removes 5, and then there is $12 - (1 + 2 + 3 + 5) = 1$ bean remaining.
On the next turn, Franco is unable to remove 4 beans and so in this case Sarah wins.

Assume that the distribution of the final 12 beans is $(11, 1)$.
Franco removes 1, and so the jars contain $(10, 1)$ or $(11, 0)$.

As we saw in the previous case, if the distribution is $(11, 0)$ at this point in the game, then Sarah wins.
So we assume that Franco removes the 1 bean from the jar containing 11 beans, so that the jars contain $(10, 1)$.
In each of the next 3 turns, beans must be removed from the jar with 10 beans: Sarah removes 2, Franco removes 3, Sarah removes 5, and then there are $10 - (2 + 3 + 5) = 0$ beans remaining in that jar and still 1 bean in the other jar.
On the next turn, Franco is unable to take 4 beans and so Sarah wins.

There are other ways that the final 12 beans can be distributed between the two jars, and for some of these distributions, Sarah is guaranteed to win.
However, to describe a winning strategy for Sarah concisely, we demonstrate how Sarah can force the distribution of the final 12 beans to be one of the two cases above, thus guaranteeing her win.

Next, we demonstrate how Sarah is able to force the distribution of the final 12 beans to be $(12, 0)$ or $(11, 1)$.
Sarah's winning strategy is to remove all beans from one of the jars, or to remove all but 1 bean from one of the jars.
Calling this jar the *target* jar, her strategy is as follows:

- If the target jar contains 5 or more beans, Sarah removes beans from the target jar on each of her turns
- If the target jar contains 2, 3 or 4 beans, Sarah removes 2 beans from the target jar (when it's her turn to remove 2), and removes 5 beans from the other jar (when it's her turn to remove 5)
- Once the target jar contains 0 beans or 1 bean, Sarah removes beans from the other jar on each of her remaining turns

Finally, we explain why Sarah is able to perform this strategy.
In each successive group of 12 turns (6 turns for Sarah), Sarah removes $3(2 + 5) = 21$ beans.
Thus in the first 109 groups of 12 turns (1308 turns), Sarah removes $109 \times 21 = 2289$ beans.
Since the jars originally contain 2022 and 2023 beans, Sarah is able to remove enough beans so that the target jar contains 0 beans or 1 bean.
After the 1308 turns, the other jar contains 12 or 11 beans, which tells us that Sarah will be able to remove the required number of beans on each of her turns.
(We note that Franco may also remove beans from the target jar, however this is okay as it has no effect on Sarah's strategy.)

Summarizing, Sarah has a strategy that will ensure that the distribution of the final 12 beans is either $(12, 0)$ or $(11, 1)$.
Given that the final 12 beans are distributed in one of these two ways, Sarah is guaranteed to win this game, and thus Sarah has a winning strategy.

It is worth noting that the player who has the winning strategy in this game, Sarah, is unique, however the argument demonstrating her guaranteed win, is not unique. For example, Sarah is also guaranteed to win if the distribution of the final 12 beans is $(10, 2)$ or $(9, 3)$, and thus this could have also be used in the description of Sarah's winning strategy.