The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2022 Euclid Contest

Tuesday, April 5, 2022
(in North America and South America)

Wednesday, April 6, 2022
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\dfrac{3^2-2^3}{2^3-3^2} = \dfrac{9-8}{8-9} = \dfrac{1}{-1} = -1$.

   Alternatively, since $2^3 - 3^2 = -(3^2-2^3)$, then $\dfrac{3^2-2^3}{2^3-3^2} = -1$.

   (b) Evaluating, $\sqrt{\sqrt{81}+\sqrt{9}-\sqrt{64}} = \sqrt{9+3-8} = \sqrt{4} = 2$.

   (c) Since $\dfrac{1}{\sqrt{x^2+7}} = \dfrac{1}{4}$, then $\sqrt{x^2+7} = 4$.

   This means that $x^2+7 = 4^2 = 16$ and so $x^2 = 9$.

   Since $x^2 = 9$, then $x = \pm 3$.

   We can check by substitution that both of these values are solutions.

2. (a) Factoring, $2022 = 2 \cdot 1011 = 2 \cdot 3 \cdot 337$. (It turns out that 337 is a prime number, though this fact is not needed here.)

   Therefore, $2022 = 2 \cdot 1011$ and $2022 = 3 \cdot 674$ and $2022 = 6 \cdot 337$.

   Thus, the three ordered pairs are $(a,b) = (2,1011), (3,674), (6,337)$.

   (b) Manipulating algebraically, the following equations are equivalent:

$$\begin{aligned} \frac{2c+1}{2d+1} &= \frac{1}{17} \\ 17(2c+1) &= 2d+1 \\ 34c+17 &= 2d+1 \\ 34c+16 &= 2d \\ d &= 17c+8 \end{aligned}$$

   Since c is an integer with $c > 0$, then $c \geq 1$, which means that $17c+8 \geq 25$.

   Therefore, the smallest possible value of d is $d = 25$.

   Note that, when $d = 25$, we obtain $c = 1$ and so $\dfrac{2c+1}{2d+1} = \dfrac{3}{51} = \dfrac{1}{17}$.

   (c) *Solution 1*

   When $x = -5$, the left side of the equation equals 0.

   This means that when $x = -5$, the right side of the equation must equal 0 as well.

   Thus, $(-5)^2 + 3(-5) + t = 0$ and so $25 - 15 + t = 0$ or $t = -10$.

   Solution 2

   Expanding the left side, we obtain

$$(px+r)(x+5) = px^2 + rx + 5px + 5r$$

   Since this is equal to $x^2 + 3x + t$ for all real numbers, then the coefficients of the two quadratic expressions must be the same.

   Comparing coefficients of x^2, we obtain $p = 1$.

   This means that

$$x^2 + rx + 5x + 5r = x^2 + 3x + t$$

   Comparing coefficients of x, we obtain $r + 5 = 3$ and so $r = -2$.

   This means that

$$x^2 + 3x - 10 = x^2 + 3x + t$$

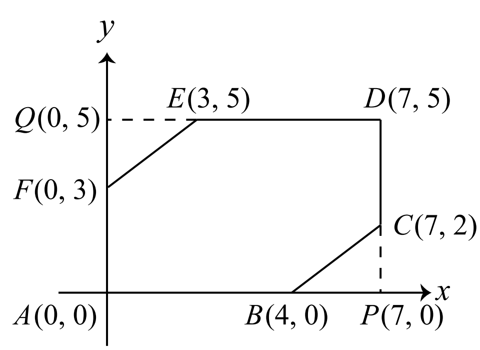
   Comparing constant terms, we obtain $t = -10$.

3. (a) Suppose that the volume of the jug is V L.
Then $\frac{1}{4}V + 24 = \frac{5}{8}V$.
Multiplying by 8, we obtain $2V + 24 \cdot 8 = 5V$ which gives $3V = 192$ and so $V = 64$.
Therefore, the volume of the jug is 64 L.

(b) Suppose that Stephanie starts with n soccer balls.
Since Stephanie can divide the n balls into fifths and into elevenths, then n is a multiple of both 5 and 11.
Since 5 and 11 are both prime numbers, then n must be a multiple of $5 \cdot 11 = 55$.
Thus, $n = 55k$ for some positive integer k.
In this case, $\frac{2}{5}n = \frac{2}{5} \cdot 55k = 22k$ and $\frac{6}{11}n = \frac{6}{11} \cdot 55k = 30k$.
When Stephanie has given these balls away, she is left with $55k - 22k - 30k = 3k$ balls.
Since $3k$ is a multiple of 9, then k is a multiple of 3.
Therefore, the smallest possible number of balls is obtained when $k = 3$, which means that Stephanie started with $n = 55 \cdot 3 = 165$ soccer balls.

(c) Suppose that the number of students in the Junior section is j and the number of students in the Senior section is s.
The number of left-handed Junior students is 60% of j, or $0.6j$.
The number of right-handed Junior students is 40% of j, or $0.4j$.
The number of left-handed Senior students is 10% of s, or $0.1s$.
The number of right-handed Senior students is 90% of s, or $0.9s$.
Since the total numbers of left-handed and right-students are equal, we obtain the equation $0.6j + 0.1s = 0.4j + 0.9s$ which gives $0.2j = 0.8s$ or $j = 4s$.
This means that there are 4 times as many Junior students as Senior students, which means that $\frac{4}{5}$ of the students are Junior and $\frac{1}{5}$ are Senior.
Therefore, 80% of the students in the math club are in the Junior section.

4. (a) Let P be the point with coordinates $(7, 0)$ and let Q be the point with coordinates $(0, 5)$.



Then $APDQ$ is a rectangle with width 7 and height 5, and so it has area $7 \cdot 5 = 35$.
Hexagon $ABCDEF$ is formed by removing two triangles from rectangle $APDQ$, namely $\triangle BPC$ and $\triangle EQF$.
Each of $\triangle BPC$ and $\triangle EQF$ is right-angled, because each shares an angle with rectangle $APDQ$.
Each of $\triangle BPC$ and $\triangle EQF$ has a base of length 3 and a height of 2.
Thus, their combined area is $2 \cdot \frac{1}{2} \cdot 3 \cdot 2 = 6$.
This means that the area of hexagon $ABCDEF$ is $35 - 6 = 29$.

(b) Since $\triangle PQS$ is right-angled at P, then by the Pythagorean Theorem,

$$SQ^2 = SP^2 + PQ^2 = (x+3)^2 + x^2$$

Since $\triangle QRS$ is right-angled at Q, then by the Pythagorean Theorem, we obtain

$$\begin{aligned} RS^2 &= SQ^2 + QR^2 \\ (x+8)^2 &= ((x+3)^2 + x^2) + 8^2 \\ x^2 + 16x + 64 &= x^2 + 6x + 9 + x^2 + 64 \\ 0 &= x^2 - 10x + 9 \\ 0 &= (x-1)(x-9) \end{aligned}$$

and so $x = 1$ or $x = 9$.
(We can check that if $x = 1$, $\triangle PQS$ has sides of lengths 4, 1 and $\sqrt{17}$ and $\triangle QRS$ has sides of lengths $\sqrt{17}$, 8 and 9, both of which are right-angled, and if $x = 9$, $\triangle PQS$ has sides of lengths 12, 9 and 15 and $\triangle QRS$ has sides of lengths 15, 8 and 17, both of which are right-angled.)
In terms of x, the perimeter of $PQRS$ is $x + 8 + (x+8) + (x+3) = 3x + 19$.
Thus, the possible perimeters of $PQRS$ are 22 (when $x = 1$) and 46 (when $x = 9$).

5. (a) If r is a term in the sequence and s is the next term, then $s = 1 + \dfrac{1}{1+r}$.

This means that $s - 1 = \dfrac{1}{1+r}$ and so $\dfrac{1}{s-1} = 1 + r$ which gives $r = \dfrac{1}{s-1} - 1$.

Therefore, since $a_3 = \dfrac{41}{29}$, then

$$a_2 = \frac{1}{a_3 - 1} - 1 = \frac{1}{(41/29) - 1} - 1 = \frac{1}{12/29} - 1 = \frac{29}{12} - 1 = \frac{17}{12}$$

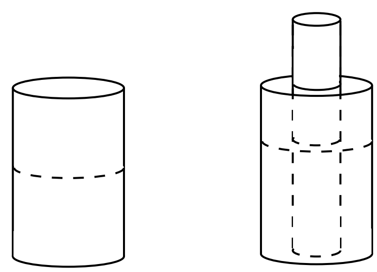
Further, since $a_2 = \dfrac{17}{12}$, then

$$a_1 = \frac{1}{a_2 - 1} - 1 = \frac{1}{(17/12) - 1} - 1 = \frac{1}{5/12} - 1 = \frac{12}{5} - 1 = \frac{7}{5}$$

(b) Initially, the water in the hollow tube forms a cylinder with radius 10 mm and height h mm. Thus, the volume of the water is $\pi(10\text{ mm})^2(h\text{ mm}) = 100\pi h\text{ mm}^3$.
After the rod is inserted, the level of the water rises to 64 mm. Note that this does not overflow the tube, since the tube's height is 100 mm.
Up to the height of the water, the tube is a cylinder with radius 10 mm and height 64 mm.
Thus, the volume of the tube up to the height of the water is

$$\pi(10\text{ mm})^2(64\text{ mm}) = 6400\pi\text{ mm}^3$$

This volume consists of the water that is in the tube (whose volume, which has not changed, is $100\pi h\text{ mm}^3$) and the rod up to a height of 64 mm.



Since the radius of the rod is 2.5 mm, the volume of the rod up to a height of 64 mm is $\pi(2.5\text{ mm})^2(64\text{ mm}) = 400\pi\text{ mm}^3$.
Comparing volumes, $6400\pi\text{ mm}^3 = 100\pi h\text{ mm}^3 + 400\pi\text{ mm}^3$ and so $100h = 6000$ which gives $h = 60$.

6. (a) We note that $\dfrac{2x+1}{x} = \dfrac{2x}{x} + \dfrac{1}{x} = 2 + \dfrac{1}{x}$.

Therefore, $\dfrac{2x+1}{x} = 4$ exactly when $2 + \dfrac{1}{x} = 4$ or $\dfrac{1}{x} = 2$ and so $x = \dfrac{1}{2}$.

Alternatively, we could solve $\dfrac{2x+1}{x} = 4$ directly to obtain $2x+1 = 4x$, which gives $2x = 1$ and so $x = \dfrac{1}{2}$.

Thus, to determine the value of $f(4)$, we substitute $x = \dfrac{1}{2}$ into the given equation $f\left(\dfrac{2x+1}{x}\right) = x + 6$ and obtain $f(4) = \dfrac{1}{2} + 6 = \dfrac{13}{2}$.

(b) Since the graph passes through $(3,5)$, $(5,4)$ and $(11,3)$, we can substitute these three points and obtain the following three equations:

$$\begin{aligned}5 &= \log_a(3+b)+c\\4 &= \log_a(5+b)+c\\3 &= \log_a(11+b)+c\end{aligned}$$

Subtracting the second equation from the first and the third equation from the second, we obtain:

$$\begin{aligned}1 &= \log_a(3+b)-\log_a(5+b)\\1 &= \log_a(5+b)-\log_a(11+b)\end{aligned}$$

Equating right sides and manipulating, we obtain the following equivalent equations:

$$\begin{aligned}\log_a(5+b)-\log_a(11+b) &= \log_a(3+b)-\log_a(5+b)\\2\log_a(5+b) &= \log_a(3+b)+\log_a(11+b)\\\log_a\left((5+b)^2\right) &= \log_a\left((3+b)(11+b)\right) \qquad \text{(using log laws)}\\(5+b)^2 &= (3+b)(11+b) \qquad \text{(raising both sides to the power of } a\text{)}\\25+10b+b^2 &= 33+14b+b^2\\-8 &= 4b\\b &= -2\end{aligned}$$

Since $b=-2$, the equation $1=\log_a(3+b)-\log_a(5+b)$ becomes $1=\log_a 1-\log_a 3$.
Since $\log_a 1=0$ for every admissible value of a, then $\log_a 3=-1$ which gives $a=3^{-1}=\frac{1}{3}$.
Finally, the equation $5=\log_a(3+b)+c$ becomes $5=\log_{1/3}(1)+c$ and so $c=5$.
Therefore, $a=\frac{1}{3}$, $b=-2$, and $c=5$, which gives $y=\log_{1/3}(x-2)+5$.

Checking:

- When $x=3$, we obtain $y=\log_{1/3}(3-2)+5=\log_{1/3}1+5=0+5=5$.
- When $x=5$, we obtain $y=\log_{1/3}(5-2)+5=\log_{1/3}3+5=-1+5=4$.
- When $x=11$, we obtain $y=\log_{1/3}(11-2)+5=\log_{1/3}9+5=-2+5=3$.

7. (a) The probability that the integer n is chosen is $\log_{100}\left(1+\dfrac{1}{n}\right)$.

The probability that an integer between 81 and 99, inclusive, is chosen equals the sum of the probabilities that the integers 81, 82, ..., 98, 99 are selected, which equals

$$\log_{100}\left(1+\frac{1}{81}\right)+\log_{100}\left(1+\frac{1}{82}\right)+\cdots+\log_{100}\left(1+\frac{1}{98}\right)+\log_{100}\left(1+\frac{1}{99}\right)$$

Since the second probability equals 2 times the first probability, the following equations are equivalent:

$$\begin{aligned}\log_{100}\left(1+\frac{1}{81}\right)+\log_{100}\left(1+\frac{1}{82}\right)+\cdots+\log_{100}\left(1+\frac{1}{98}\right)+\log_{100}\left(1+\frac{1}{99}\right)&=2\log_{100}\left(1+\frac{1}{n}\right)\\ \log_{100}\left(\frac{82}{81}\right)+\log_{100}\left(\frac{83}{82}\right)+\cdots+\log_{100}\left(\frac{99}{98}\right)+\log_{100}\left(\frac{100}{99}\right)&=2\log_{100}\left(1+\frac{1}{n}\right)\end{aligned}$$

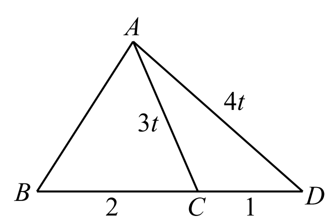
Using logarithm laws, these equations are further equivalent to

$$\begin{aligned}\log_{100}\left(\frac{82}{81}\cdot\frac{83}{82}\cdot\cdots\cdot\frac{99}{98}\cdot\frac{100}{99}\right)&=\log_{100}\left(1+\frac{1}{n}\right)^2\\ \log_{100}\left(\frac{100}{81}\right)&=\log_{100}\left(1+\frac{1}{n}\right)^2\end{aligned}$$

Since logarithm functions are invertible, we obtain $\dfrac{100}{81}=\left(1+\dfrac{1}{n}\right)^2$.

Since $n>0$, then $1+\dfrac{1}{n}=\sqrt{\dfrac{100}{81}}=\dfrac{10}{9}$, and so $\dfrac{1}{n}=\dfrac{1}{9}$, which gives $n=9$.

(b) Since $\dfrac{AC}{AD}=\dfrac{3}{4}$, then we let $AC=3t$ and $AD=4t$ for some real number $t>0$.



Using the cosine law in $\triangle ACD$, the following equations are equivalent:

$$\begin{aligned}AD^2&=AC^2+CD^2-2\cdot AC\cdot CD\cdot\cos(\angle ACD)\\ (4t)^2&=(3t)^2+1^2-2(3t)(1)(-\tfrac{3}{5})\\ 16t^2&=9t^2+1+\tfrac{18}{5}t\\ 80t^2&=45t^2+5+18t\\ 35t^2-18t-5&=0\\ (7t-5)(5t+1)&=0\end{aligned}$$

Since $t>0$, then $t=\frac{5}{7}$.

Thus, $AC=3t=\frac{15}{7}$.

Using the cosine law in $\triangle ACB$ and noting that

$$\cos(\angle ACB)=\cos(180^\circ-\angle ACD)=-\cos(\angle ACD)=\tfrac{3}{5}$$

the following equations are equivalent:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2 \cdot AC \cdot BC \cdot \cos(\angle ACB) \\ &= \left(\tfrac{15}{7}\right)^2 + 2^2 - 2(\tfrac{15}{7})(2)(\tfrac{3}{5}) \\ &= \tfrac{225}{49} + 4 - \tfrac{36}{7} \\ &= \tfrac{225}{49} + \tfrac{196}{49} - \tfrac{252}{49} \\ &= \tfrac{169}{49} \end{aligned}$$

Since $AB > 0$, then $AB = \frac{13}{7}$.

8. (a) The parabola with equation $y = ax^2 + 2$ is symmetric about the y-axis.
Thus, its vertex occurs when $x = 0$ (which gives $y = a \cdot 0^2 + 2 = 2$) and so V has coordinates $(0, 2)$.
To find the coordinates of B and C, we use the equations of the parabola and line to obtain

$$\begin{aligned} ax^2 + 2 &= -x + 4a \\ ax^2 + x + (2 - 4a) &= 0 \end{aligned}$$

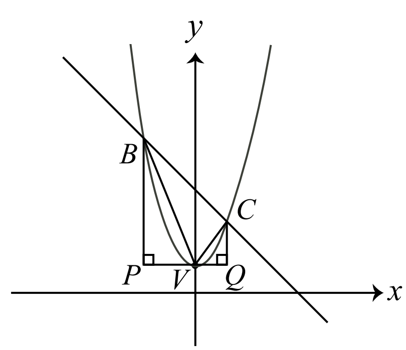
Using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1^2 - 4a(2-4a)}}{2a} = \frac{-1 \pm \sqrt{1 - 8a + 16a^2}}{2a}$$

Since $1 - 8a + 16a^2 = (4a-1)^2$ and $4a - 1 > 0$ (since $a > \frac{1}{2}$), then $\sqrt{1 - 8a + 16a^2} = 4a - 1$ and so

$$x = \frac{-1 \pm (4a-1)}{2a}$$

which means that $x = \dfrac{4a-2}{2a} = \dfrac{2a-1}{a} = 2 - \dfrac{1}{a}$ or $x = \dfrac{-4a}{2a} = -2$.
We can use the equation of the line to obtain the y-coordinates of B and C.
When $x = -2$ (corresponding to point B), we obtain $y = -(-2) + 4a = 4a + 2$.
When $x = 2 - \dfrac{1}{a}$ (corresponding to point C), we obtain $y = -\left(2 - \dfrac{1}{a}\right) + 4a = 4a - 2 + \dfrac{1}{a}$.
Let P and Q be the points on the horizontal line through V so that BP and CQ are perpendicular to PQ.



Then the area of $\triangle VBC$ is equal to the area of trapezoid $PBCQ$ minus the areas of right-angled $\triangle BPV$ and right-angled $\triangle CQV$.
Since B has coordinates $(-2, 4a+2)$, P has coordinates $(-2, 2)$, V has coordiantes $(0, 2)$, Q has coordinates $\left(2-\dfrac{1}{a}, 2\right)$, and C has coordinates $\left(2-\dfrac{1}{a}, 4a-2+\dfrac{1}{a}\right)$, then

$$\begin{aligned}
BP &= (4a+2)-2 = 4a \\
CQ &= \left(4a-2+\frac{1}{a}\right)-2 = 4a-4+\frac{1}{a} \\
PV &= 0-(-2) = 2 \\
QV &= 2-\frac{1}{a}-0 = 2-\frac{1}{a} \\
PQ &= PV+QV = 2+2-\frac{1}{a} = 4-\frac{1}{a}
\end{aligned}$$

Therefore, the area of trapezoid $PBCQ$ is

$$\frac{1}{2}(BP+CQ)(PQ) = \frac{1}{2}\left(4a+4a-4+\frac{1}{a}\right)\left(4-\frac{1}{a}\right) = \left(4a-2+\frac{1}{2a}\right)\left(4-\frac{1}{a}\right)$$

Also, the area of $\triangle BPV$ is $\dfrac{1}{2}\cdot BP\cdot PV = \dfrac{1}{2}(4a)(2) = 4a$.
Furthermore, the area of $\triangle CQV$ is

$$\frac{1}{2}\cdot CQ\cdot QV = \frac{1}{2}\left(4a-4+\frac{1}{a}\right)\left(2-\frac{1}{a}\right) = \left(2a-2+\frac{1}{2a}\right)\left(2-\frac{1}{a}\right)$$

From the given information,

$$\left(4a-2+\frac{1}{2a}\right)\left(4-\frac{1}{a}\right)-4a-\left(2a-2+\frac{1}{2a}\right)\left(2-\frac{1}{a}\right) = \frac{72}{5}$$

Multiplying both sides by $2a^2$, which we distribute through the factors on the left side as $2a\cdot a$, we obtain

$$(8a^2-4a+1)(4a-1)-8a^3-(4a^2-4a+1)(2a-1) = \frac{144}{5}a^2$$

Multiplying both sides by 5, we obtain

$$5(8a^2-4a+1)(4a-1)-40a^3-5(4a^2-4a+1)(2a-1) = 144a^2$$

Expanding and simplifying, we obtain

$$\begin{aligned}
(160a^3-120a^2+40a-5)-40a^3-(40a^3-60a^2+30a-5) &= 144a^2 \\
80a^3-204a^2+10a &= 0 \\
2a(40a^2-102a+5) &= 0 \\
2a(20a-1)(2a-5) &= 0
\end{aligned}$$

and so $a=0$ or $a=\dfrac{1}{20}$ or $a=\dfrac{5}{2}$. Since $a>\dfrac{1}{2}$, then $a=\dfrac{5}{2}$.

(b) We prove that there cannot be such a triangle.

We prove this by contradiction. That is, we suppose that there is such a triangle and prove that there is then a logical contradiction.

Suppose that $\triangle ABC$ is not equilateral, has side lengths that form a geometric sequence, and angles whose measures form an arithmetic sequence.

Suppose that $\triangle ABC$ has side lengths $BC = a$, $AC = ar$, and $AB = ar^2$, for some real numbers $a > 0$ and $r > 1$. (These lengths form a geometric sequence, and we can assume that this sequence is increasing, and that the sides are labelled in this particular order.)

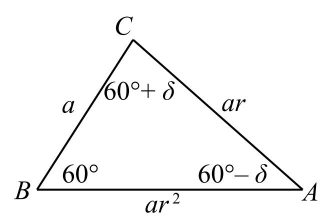
Since $BC < AC < AB$, then the opposite angles have the same relationships, namely $\angle BAC < \angle ABC < \angle ACB$.

Suppose that $\angle BAC = \theta$, $\angle ABC = \theta + \delta$, and $\angle ACB = \theta + 2\delta$ for some angles θ and δ. (In other words, these angles form an arithmetic sequence.

Since these three angles are the angles in a triangle, then their sum is 180°, and so

$$\begin{aligned} \theta + (\theta + \delta) + (\theta + 2\delta) &= 180^\circ \\ 3\theta + 3\delta &= 180^\circ \\ \theta + \delta &= 60^\circ \end{aligned}$$

In other words, $\angle ABC = 60^\circ$.



We could proceed using the cosine law:

$$\begin{aligned} AC^2 &= BC^2 + AB^2 - 2 \cdot BC \cdot AB \cdot \cos(\angle ABC) \\ (ar)^2 &= a^2 + (ar^2)^2 - 2a(ar^2)\cos(60^\circ) \\ a^2r^2 &= a^2 + a^2r^4 - 2a^2r^2 \cdot \tfrac{1}{2} \\ a^2r^2 &= a^2 + a^2r^4 - a^2r^2 \\ 0 &= a^2r^4 - 2a^2r^2 + a^2 \\ 0 &= a^2(r^4 - 2r^2 + 1) \\ 0 &= a^2(r^2 - 1)^2 \end{aligned}$$

This tells us that $a = 0$ (which is impossible) or $r^2 = 1$ (and thus $r = \pm 1$, which is impossible).

Therefore, we have reached a logical contradiction and so such a triangle cannot exist.

Alternatively, we could proceed using the sine law, noting that

$$\begin{aligned} \angle BAC &= \theta = (\theta + \delta) - \delta = 60^\circ - \delta \\ \angle ACB &= \theta + 2\delta = (\theta + \delta) + \delta = 60^\circ + \delta \end{aligned}$$

By the sine law,

$$\frac{BC}{\sin(\angle BAC)} = \frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle ACB)}$$

from which we obtain

$$\frac{a}{\sin(60^\circ - \delta)} = \frac{ar}{\sin(60^\circ)} = \frac{ar^2}{\sin(60^\circ + \delta)}$$

Since $a \neq 0$, from the first two parts,

$$r = \frac{ar}{a} = \frac{\sin 60^\circ}{\sin(60^\circ - \delta)}$$

Since $ar \neq 0$, from the second two parts,

$$r = \frac{ar^2}{ar} = \frac{\sin(60^\circ + \delta)}{\sin 60^\circ}$$

Equating expressions for r, we obtain successively

$$\begin{aligned}
\frac{\sin 60^\circ}{\sin(60^\circ - \delta)} &= \frac{\sin(60^\circ + \delta)}{\sin 60^\circ} \\
\sin^2 60^\circ &= \sin(60^\circ - \delta)\sin(60^\circ + \delta) \\
\left(\tfrac{\sqrt{3}}{2}\right)^2 &= (\sin 60^\circ \cos\delta - \cos 60^\circ \sin\delta)(\sin 60^\circ \cos\delta + \cos 60^\circ \sin\delta) \\
\tfrac{3}{4} &= \left(\tfrac{\sqrt{3}}{2}\cos\delta - \tfrac{1}{2}\sin\delta\right)\left(\tfrac{\sqrt{3}}{2}\cos\delta + \tfrac{1}{2}\sin\delta\right) \\
\tfrac{3}{4} &= \tfrac{3}{4}\cos^2\delta - \tfrac{1}{4}\sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4}\cos^2\delta + \tfrac{3}{4}\sin^2\delta - \sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4}(\cos^2\delta + \sin^2\delta) - \sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4} - \sin^2\delta \\
\sin^2\delta &= 0
\end{aligned}$$

which means that $\delta = 0^\circ$. (Any other angle δ with $\sin\delta = 0$ would not produce angles in a triangle.)

Therefore, all three angles in the triangle are 60°, which means that the triangle is equilateral, which it cannot be.

Therefore, we have reached a logical contradiction and so such a triangle cannot exist.

9. (a) The $(4,2)$-sawtooth sequence consists of the terms

$$1, \quad 2,3,4,3,2,1, \quad 2,3,4,3,2,1$$

whose sum is 31.

(b) *Solution 1*

Suppose that $m \geq 2$.
The $(m,3)$-sawtooth sequence consists of an initial 1 followed by 3 teeth, each of which goes from 2 to m to 1.
Consider one of these teeth whose terms are

$$2,3,4,\ldots,m-1,m,m-1,m-2,m-3,\ldots,2,1$$

When we write the ascending portion directly above the descending portion, we obtain

$$\begin{array}{cccccc} 2, & 3, & 4, & \ldots, & m-1, & m, \\ m-1, & m-2, & m-3, & \ldots, & 2, & 1 \end{array}$$

From this presentation, we can see $m-1$ pairs of terms, the sum of each of which is $m+1$. (Note that $2+(m-1) = 3+(m-2) = 4+(m-3) = \cdots = (m-1)+2 = m+1$ and as we move from left to right, the terms on the top increase by 1 at each step and the terms on the bottom decrease by 1 at each step, so their sum is indeed constant.)
Therefore, the sum of the numbers in one of the teeth is $(m-1)(m+1) = m^2-1$.
This means that the sum of the terms in the $(m,3)$-sawtooth sequence is $1+3(m^2-1)$, which equals $3m^2-2$.

Solution 2

Suppose that $m \geq 2$.
The $(m,3)$-sawtooth sequence consists of an initial 1 followed by 3 teeth, each of which goes from 2 to m to 1.
Consider one of these teeth whose terms are

$$2,3,4,\ldots,m-1,m,m-1,m-2,m-3,\ldots,2,1$$

This tooth includes one 1, two 2s, two 3s, and so on, until we reach two $(m-1)$s, and one m.
The sum of these numbers is

$$1(1)+2(2)+2(3)+\cdots+2(m-1)+m$$

which can be rewritten as

$$2(1+2+3+\cdots+(m-1)+m)-1-m = 2\cdot\tfrac{1}{2}m(m+1)-m-1 = m^2+m-m-1 = m^2-1$$

Therefore, the sum of the numbers in one of the teeth is $(m-1)(m+1) = m^2-1$.
This means that the sum of the terms in the $(m,3)$-sawtooth sequence is $1+3(m^2-1)$, which equals $3m^2-2$.

(c) From (b), the sum of the terms in each tooth is $m^2 - 1$.
Thus, the sum of the terms in the (m, n)-sawtooth sequence is $1 + n(m^2 - 1)$.
For this to equal 145, we have $n(m^2 - 1) = 144$.
This means that n and $m^2 - 1$ form a divisor pair of 144.
As m ranges from 2 to 12, the values of $m^2 - 1$ are

$$3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143$$

(When $m = 13$, we get $m^2 - 1 = 168$ and so when $m \geq 13$, the value of $m^2 - 1$ is too large to be a divisor of 144.)
Of these, $3, 8, 24, 48$ are divisors of 144 (corresponding to $m = 2, 3, 5, 7$), and give corresponding divisors $48, 18, 6, 3$.
Therefore, the pairs (m, n) for which the sum of the terms is 145 are

$$(m, n) = (2, 48), (3, 18), (5, 6), (7, 3)$$

(d) In an (m, n)-sawtooth sequence, the sum of the terms is $n(m^2 - 1) + 1$.
In each tooth, there are $(m - 1) + (m - 1) = 2m - 2$ terms (from 2 to m, inclusive, and from $m - 1$ to 1, inclusive).
This means that there are $n(2m - 2) + 1$ terms in the sequence.
Thus, the average of the terms in the sequence is $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1}$.
We need to prove that this is not an integer for all pairs of positive integers (m, n) with $m \geq 2$.
Suppose that $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1} = k$ for some integer k. We will show, by contradiction, that this is not possible.
Since $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1} = k$, then

$$\begin{aligned}
\frac{m^2 n - n + 1}{2mn - 2n + 1} &= k \\
m^2 n - n + 1 &= 2mnk - 2nk + k \\
m^2 n - 2mnk + (2nk - n - k + 1) &= 0
\end{aligned}$$

We treat this as a quadratic equation in m.
Since m is an integer, then this equation has integer roots, and so its discriminant must be a perfect square.
The discriminant of this quadratic equation is

$$\begin{aligned}
\Delta &= (-2nk)^2 - 4n(2nk - n - k + 1) \\
&= 4n^2k^2 - 8n^2k + 4n^2 + 4nk - 4n \\
&= 4n^2(k^2 - 2k + 1) + 4n(k - 1) \\
&= 4n^2(k - 1)^2 + 4n(k - 1) \\
&= (2n(k - 1))^2 + 2(2n(k - 1)) + 1 - 1 \\
&= (2n(k - 1) + 1)^2 - 1
\end{aligned}$$

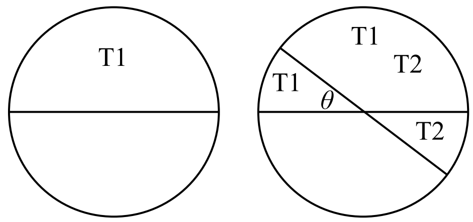
We note that $(2n(k-1)+1)^2$ is a perfect square and Δ is supposed to be a perfect square.
But these perfect squares differ by 1, and the only two perfect squares that differ by 1 are

1 and 0.
(To justify this last fact, we could look at the equation $a^2 - b^2 = 1$ where a and b are non-negative integers, and factor this to obtain $(a+b)(a-b) = 1$ which would give $a+b = a-b = 1$ from which we get $a = 1$ and $b = 0$.)
Since $(2n(k-1)+1)^2 = 1$ and $2n(k-1)+1$ is non-negative, then $2n(k-1)+1 = 1$ and so $2n(k-1) = 0$.
Since n is positive, then $k-1 = 0$ or $k = 1$.
Therefore, the only possible way in which the average is an integer is if the average is 1.
In this case, we get

$$\begin{aligned} m^2n - 2mn + (2n - n - 1 + 1) &= 0 \\ m^2n - 2mn + n &= 0 \\ n(m^2 - 2m + 1) &= 0 \\ n(m-1)^2 &= 0 \end{aligned}$$

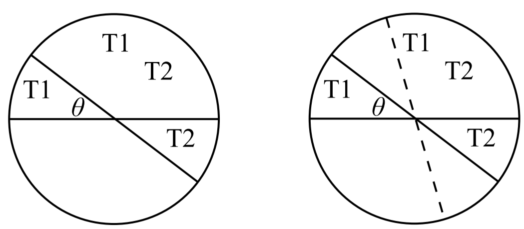
Since n and m are positive integers with $m \geq 2$, then $n(m-1)^2 \neq 0$, which is a contradiction.
Therefore, the average of the terms in an (m,n)-sawtooth sequence cannot be an integer.

10. (a) Assume that the first topping is placed on the top half of the pizza. (We can rotate the pizza so that this is the case.)
Assume that the second topping is placed on the half of the pizza that is above the horizontal diameter that makes an angle of θ clockwise with the horizontal as shown. In other words, the topping covers the pizza from θ to $\theta + 180°$.



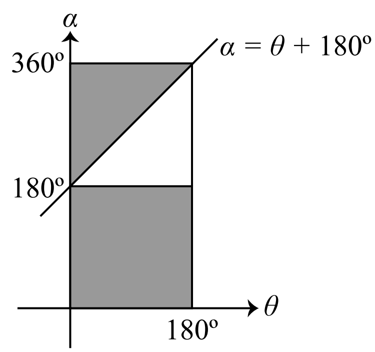
We may assume that $0° \leq \theta \leq 360°$.
When $0° \leq \theta \leq 90°$, the angle of the sector covered by both toppings is at least $90°$ (and so is at least a quarter of the circle).
When $90° < \theta \leq 180°$, the angle of the sector covered by both toppings is less than $90°$ (and so is less than a quarter of the circle).
When θ moves past $180°$, the left-hand portion of the upper half circle starts to be covered with both toppings again. When $180° \leq \theta < 270°$, the angle of the sector covered by both toppings is less than $90°$ (and so is less than a quarter of the circle).
When $270° \leq \theta \leq 360°$, the angle of the sector covered by both toppings at least $90°$ (and so is at least a quarter of the circle).
Therefore, if θ is chosen randomly between $0°$ and $360°$, the combined length of the intervals in which at least $\frac{1}{4}$ of the pizza is covered with both toppings is $180°$.
Therefore, the probability is $\dfrac{180°}{360°}$, or $\dfrac{1}{2}$.

(b) Suppose that the first topping is placed on the top half of the pizza. (Again, we can rotate the pizza so that this is the case.)
Assume that the second topping is placed on the half of the pizza that is above the diameter that makes an angle of θ clockwise with the horizontal as shown. In other words, the topping covers the pizza from θ to $\theta + 180°$.
We may assume that $0° \leq \theta \leq 180°$. If $180° \leq \theta \leq 360°$, the resulting pizza can be seen as a reflection of the one shown.



Consider the third diameter added, shown dotted in the diagram above. Suppose that its angle with the horizontal is α. (In the diagram, $\alpha < 90°$.) We assume that the topping is added on the half pizza clockwise beginning at the angle of α, and that this topping stays in the same relative position as the diameter sweeps around the circle.
For what angles α will there be a portion of the pizza covered with all three toppings?
If $0° \leq \alpha < 180°$, there will be a portion covered with three toppings; this portion is above the right half of the horizontal diameter.
If $180° \leq \alpha < 180° + \theta$, the third diameter will pass through the two regions with angle θ and the third topping will be below this diameter, so there will not be a region covered

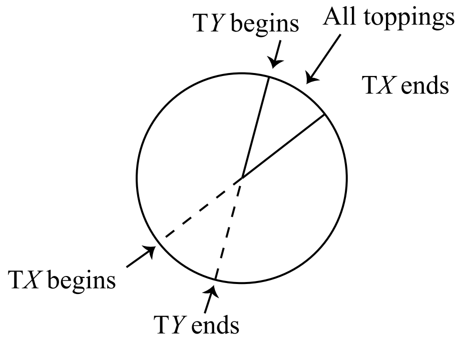
with three toppings.
If $180^\circ + \theta \leq \alpha \leq 360^\circ$, the third topping starts to cover the leftmost part of the region currently covered with two toppings, and so a region is covered with three toppings.
Therefore, for an angle θ with $0^\circ \leq \theta \leq 180^\circ$, a region of the pizza is covered with three toppings when $0^\circ \leq \alpha < 180^\circ$ and when $180^\circ + \theta \leq \alpha \leq 360^\circ$.
To determine the desired probability, we graph points (θ, α). A particular choice of diameters corresponds to a choice of angles θ and α with $0^\circ \leq \theta \leq 180^\circ$ and $0^\circ \leq \alpha \leq 360^\circ$, which corresponds to a point on the graph below.
The probability that we are looking for then equals the area of the region of this graph where three toppings are in a portion of the pizza divided by the total allowable area of the graph.
The shaded region of the graph corresponds to instances where a portion of the pizza will be covered by three toppings.



This shaded region consists of the entire portion of the graph where $0^\circ \leq \alpha \leq 180^\circ$ (regardless of θ) as well as the region above the line with equation $\alpha = \theta + 180^\circ$ (that is, the region with $\theta + 180^\circ \leq \alpha \leq 360^\circ$).
Since the slope of the line is 1, it divides the upper half of the region, which is a square, into two pieces of equal area.
Therefore, $\frac{3}{4}$ of the graph is shaded, which means that the probability that a region of the pizza is covered by all three toppings is $\dfrac{3}{4}$.

(c) The main idea of this solution is that the toppings all overlap exactly when there is one topping with the property that all other toppings "begin" somewhere in that toppings semi-circle. In the rest of this solution, we determine the probability using this fact and then justify this fact.

Suppose that, for $1 \leq j \leq N$, topping j is put on the semi-circle that starts at an angle of θ_j clockwise from the horizontal left-hand radius and continues to an angle of $\theta_j + 180^\circ$, where $0^\circ \leq \theta_j < 360^\circ$. By establishing these variables and this convention, we are fixing both the angle of the diameter and the semi-circle defined by this diameter on which the topping is placed.
Suppose that there is some region of the pizza with non-zero area that is covered by all N toppings.
This region will be a sector with two bounding radii, each of which must be half of a diameter that defines one of the toppings.
Suppose that the radius at the clockwise "end" of the sector is the end of the semi-circle where topping X is placed, and that the radius at the counter-clockwise "beginning" of the sector is the start of the semi-circle where topping Y is placed.



This means that each of the other $N-2$ toppings begins between (in the clockwise sense) the points where topping X begins and where topping Y begins.
Consider the beginning angle for topping X, θ_X.
To say that the other $N-1$ toppings begin at some point before topping X ends is the same as saying that each θ_j with $j \neq X$ is between θ_X and $\theta_X + 180^\circ$.
Here, we can allow for the possibility that $\theta_X + 180^\circ$ is greater than 360° by saying that an angle equivalent to θ_j (which is either θ_j or $\theta_j + 360^\circ$) is between θ_X and $\theta_X + 180^\circ$.
For each $j \neq X$, the angle θ_j is randomly, uniformly and independently chosen on the circle, so there is a probability of $\frac{1}{2}$ that this angle (or its equivalent) will be in the semi-circle between θ_X and $\theta_X + 180^\circ$.
Since there are $N-1$ such angles, the probability that all are between θ_X and $\theta_X + 180^\circ$ is $\frac{1}{2^{N-1}}$.
Since there are N possible selections for the first topping that can end the common sector, then the desired probability will be $\frac{N}{2^{N-1}}$ as long as we can show that no set of angles can give two different sectors that are both covered with all toppings.
To show this last fact, we suppose without loss of generality that

$$0^\circ = \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_{N-1} < \theta_N < 180^\circ$$

(We can relabel the toppings if necessary to obtain this order and rotate the pizza so that topping 1 begins at 0°.)
We need to show that it is not possible to have a Z for which $\theta_Z, \theta_{Z+1}, \ldots, \theta_N, \theta_1, \theta_2, \ldots, \theta_{Z-1}$ all lie in a semi-circle starting with θ_Z.
Since $\theta_Z < 180^\circ$ and θ_1 can be thought of as 360°, then this is not possible as θ_1 and the angles after it are all not within 180° of θ_Z.
Therefore, it is not possible to have two such regions with the same set of angles, and so the desired probability is $\frac{N}{2^{N-1}}$.