The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 5, 2022
(in North America and South America)

Wednesday, April 6, 2022
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the value of $\dfrac{3^2 - 2^3}{2^3 - 3^2}$?

   (b) What is the value of $\sqrt{\sqrt{81} + \sqrt{9} - \sqrt{64}}$?

   (c) Determine all real numbers x for which $\dfrac{1}{\sqrt{x^2 + 7}} = \dfrac{1}{4}$.

2. (a) Find the three ordered pairs of integers (a, b) with $1 < a < b$ and $ab = 2022$.

   (b) Suppose that c and d are integers with $c > 0$ and $d > 0$ and $\dfrac{2c + 1}{2d + 1} = \dfrac{1}{17}$. What is the smallest possible value of d?

   (c) Suppose that p, r and t are real numbers for which $(px + r)(x + 5) = x^2 + 3x + t$ is true for all real numbers x. Determine the value of t.

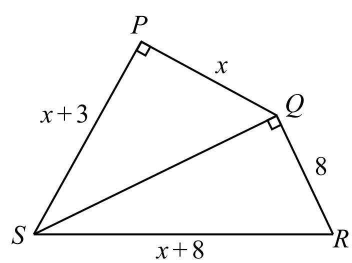
3. (a) A large water jug is $\frac{1}{4}$ full of water. After 24 litres of water are added, the jug is $\frac{5}{8}$ full. What is the volume of the jug, in litres?

(b) Stephanie starts with a large number of soccer balls. She gives $\frac{2}{5}$ of them to Alphonso and $\frac{6}{11}$ of them to Christine. The number of balls that she is left with is a multiple of 9. What is the smallest number of soccer balls with which Stephanie could have started?

(c) Each student in a math club is in either the Junior section or the Senior section. No student is in both sections.
Of the Junior students, 60% are left-handed and 40% are right-handed.
Of the Senior students, 10% are left-handed and 90% are right-handed.
No student in the math club is both left-handed and right-handed.
The total number of left-handed students is equal to the total number of right-handed students in the math club.
Determine the percentage of math club members that are in the Junior section.

4. (a) Hexagon $ABCDEF$ has vertices $A(0,0)$, $B(4,0)$, $C(7,2)$, $D(7,5)$, $E(3,5)$, $F(0,3)$. What is the area of hexagon $ABCDEF$?

(b) In the diagram, $\triangle PQS$ is right-angled at P and $\triangle QRS$ is right-angled at Q. Also, $PQ = x$, $QR = 8$, $RS = x + 8$, and $SP = x + 3$ for some real number x. Determine all possible values of the perimeter of quadrilateral $PQRS$.



5. (a) A list a_1, a_2, a_3, a_4 of rational numbers is defined so that if one term is equal to r, then the next term is equal to $1 + \dfrac{1}{1+r}$. For example, if $a_3 = \dfrac{41}{29}$, then $a_4 = 1 + \dfrac{1}{1 + (41/29)} = \dfrac{99}{70}$. If $a_3 = \dfrac{41}{29}$, what is the value of a_1?

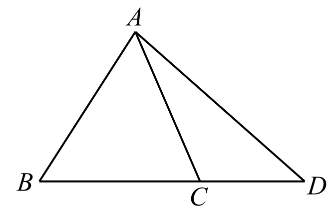
(b) A hollow cylindrical tube has a radius of 10 mm and a height of 100 mm. The tube sits flat on one of its circular faces on a horizontal table. The tube is filled with water to a depth of h mm. A solid cylindrical rod has a radius of 2.5 mm and a height of 150 mm. The rod is inserted into the tube so that one of its circular faces sits flat on the bottom of the tube. The height of the water in the tube is now 64 mm. Determine the value of h.

6. (a) A function f has the property that $f\left(\frac{2x+1}{x}\right) = x + 6$ for all real values of $x \neq 0$. What is the value of $f(4)$?

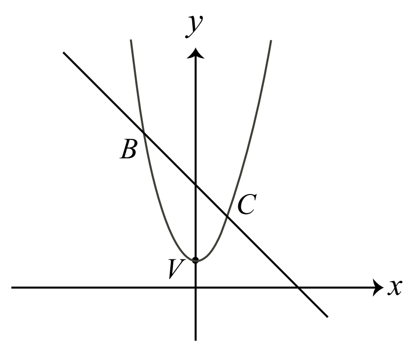
(b) Determine all real numbers a, b and c for which the graph of the function $y = \log_a(x + b) + c$ passes through the points $P(3, 5)$, $Q(5, 4)$ and $R(11, 3)$.

7. (a) A computer is programmed to choose an integer between 1 and 99, inclusive, so that the probability that it selects the integer x is equal to $\log_{100}\left(1 + \frac{1}{x}\right)$. Suppose that the probability that $81 \leq x \leq 99$ is equal to 2 times the probability that $x = n$ for some integer n. What is the value of n?

(b) In the diagram, $\triangle ABD$ has C on BD. Also, $BC = 2$, $CD = 1$, $\frac{AC}{AD} = \frac{3}{4}$, and $\cos(\angle ACD) = -\frac{3}{5}$. Determine the length of AB.



8. (a) Suppose that $a > \frac{1}{2}$ and that the parabola with equation $y = ax^2 + 2$ has vertex V. The parabola intersects the line with equation $y = -x + 4a$ at points B and C, as shown. If the area of $\triangle VBC$ is $\frac{72}{5}$, determine the value of a.



(b) Consider the following statement:

> There is a triangle that is not equilateral whose side lengths form a geometric sequence, and the measures of whose angles form an arithmetic sequence.

Show that this statement is true by finding such a triangle or prove that it is false by demonstrating that there cannot be such a triangle.

9. Suppose that m and n are positive integers with $m \geq 2$. The (m, n)-*sawtooth sequence* is a sequence of consecutive integers that starts with 1 and has n teeth, where each tooth starts with 2, goes up to m and back down to 1. For example, the $(3, 4)$-sawtooth sequence is

$$\begin{array}{ccccccccccccccccc}
 & & 3 & & & & 3 & & & & 3 & & & & 3 & & \\
 & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & \\
1 & & & & 1 & & & & 1 & & & & 1 & & & & 1
\end{array}$$

The $(3, 4)$-sawtooth sequence includes 17 terms and the average of these terms is $\frac{33}{17}$.

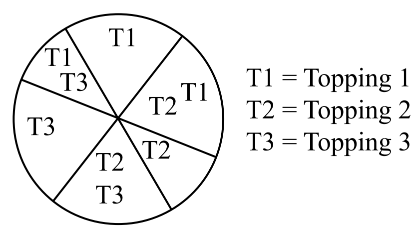
(a) Determine the sum of the terms in the $(4, 2)$-sawtooth sequence.

(b) For each positive integer $m \geq 2$, determine a simplified expression for the sum of the terms in the $(m, 3)$-sawtooth sequence.

(c) Determine all pairs (m, n) for which the sum of the terms in the (m, n)-sawtooth sequence is 145.

(d) Prove that, for all pairs of positive integers (m, n) with $m \geq 2$, the average of the terms in the (m, n)-sawtooth sequence is not an integer.

10. At Pizza by Alex, toppings are put on circular pizzas in a random way. Every topping is placed on a randomly chosen semi-circular half of the pizza and each topping's semi-circle is chosen independently. For each topping, Alex starts by drawing a diameter whose angle with the horizonal is selected uniformly at random. This divides the pizza into two semi-circles. One of the two halves is then chosen at random to be covered by the topping.



(a) For a 2-topping pizza, determine the probability that at least $\frac{1}{4}$ of the pizza is covered by both toppings.

(b) For a 3-topping pizza, determine the probability that some region of the pizza with non-zero area is covered by all 3 toppings. (The diagram above shows an example where no region is covered by all 3 toppings.)

(c) Suppose that N is a positive integer. For an N-topping pizza, determine the probability, in terms of N, that some region of the pizza with non-zero area is covered by all N toppings.

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For students...

Thank you for writing the 2022 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2022 Canadian Senior Mathematics Contest, which will be written in November 2022.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2022/2023 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results