The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2022 Cayley Contest

(Grade 10)

Wednesday, February 23, 2022
(in North America and South America)

Thursday, February 24, 2022
(outside of North America and South America)

Solutions

1. Evaluating, $2 + (0 \times 2^2) = 2 + 0 = 2$.

   Answer: (B)

2. The ones digit of 119 is not even, so 119 is not a multiple of 2.
   The ones digit of 119 is not 0 or 5, so 119 is not a multiple of 5.
   Since $120 = 3 \times 40$, then 119 is 1 less than a multiple of 3 so is not itself a multiple of 3.
   Since $110 = 11 \times 10$ and $121 = 11 \times 11$, then 119 is between two consecutive multiples of 11, so is not itself a multiple of 11.
   Finally, $119 \div 7 = 17$, so 119 is a multiple of 7.

   Answer: (D)

3. The fractions $\frac{3}{10}$ and $\frac{5}{23}$ are each less than $\frac{1}{2}$ (which is choice (E)) so cannot be the greatest among the choices. (Note that $\frac{1}{2} = \frac{5}{10} = \frac{11.5}{23}$ which we can use to compare the given fractions to $\frac{1}{2}$.)
   The fractions $\frac{4}{7}$ and $\frac{2}{3}$ are each greater than $\frac{1}{2}$, so $\frac{1}{2}$ cannot be the greatest among the choices.
   This means that the answer must be either $\frac{4}{7}$ or $\frac{2}{3}$.
   Using a common denominator of $3 \times 7 = 21$, we re-write these fractions as $\frac{4}{7} = \frac{12}{21}$ and $\frac{2}{3} = \frac{14}{21}$ which shows that $\frac{2}{3}$ has the gretest value among the five choices.
   (Alternatively, we could have converted the fractions into decimals and used their decimal approximations to compare their sizes.)

   Answer: (D)

4. The sequence consists of a pattern of 5 shapes that are repeated.
   The first repetitions of this pattern end on the 5th, 10th, 15th, 20th, and 25th shapes.
   This means that the 22nd shape is the 2nd shape after the 20th shape, and so is the 2nd shape in the pattern.
   Thus, the 22nd shape is $\square$.

   Answer: (A)

5. The given sum includes 5 terms each equal to (5×5).
   Thus, the given sum is equal to $5 \times (5 \times 5)$ which equals 5×25 or 125.

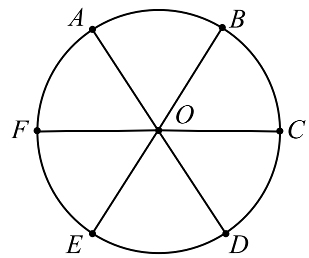
   Answer: (E)

6. Yihana is walking uphill exactly when the graph is increasing (that is, when the slope of the segment of the graph is positive).
   This is between 0 and 3 minutes and between 8 and 10 minutes, which correspond to lengths of time of 3 minutes and 2 minutes in these two cases, for a total of 5 minutes.

   Answer: (A)

7. *Solution 1*
   Since A, B, C, D, E, and F are equally spaced around the circle, moving from one point to the next corresponds to moving $\frac{1}{6}$ of the way around the circle.
   Therefore, moving from A to C corresponds to moving $\frac{2}{6}$ (or $\frac{1}{3}$) of the way around the circle.
   Since moving around the whole circle corresponds to moving through 360°, then moving $\frac{1}{3}$ of the way around the circle corresponds to moving through $\frac{1}{3} \times 360^\circ = 120^\circ$.
   Thus, $\angle AOC = 120^\circ$.

Solution 2
We join A, B, C, D, E, and F to O.



Since A, B, C, D, E, and F are equally spaced around the circle, the angles made at the centre by consecutive points are equal. That is,

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle FOA$$

Since these 6 angles form a complete circle, the sum of their measures is 360°.
Therefore,

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle FOA = \tfrac{1}{6} \times 360^\circ = 60^\circ$$

This means that $\angle AOC = \angle AOB + \angle BOC = 60^\circ + 60^\circ = 120^\circ$.

Answer: (D)

8. Since the rectangle has positive integer side lengths and an area of 24, its length and width must be a positive divisor pair of 24.
Therefore, the length and width must be 24 and 1, or 12 and 2, or 8 and 3, or 6 and 4.
Since the perimeter of a rectangle equals 2 times the sum of the length and width, the possible perimeters are

$$2(24+1) = 50 \qquad 2(12+2) = 28 \qquad 2(8+3) = 22 \qquad 2(6+4) = 20$$

These all appear as choices, which means that the perimeter of the rectangle cannot be 36, which is (E).

Answer: (E)

9. Using the definition, $3\nabla b = \dfrac{3+b}{3-b}$.
Assuming $b \neq 3$, the following equations are equivalent:

$$\begin{aligned} 3\nabla b &= -4 \\ \frac{3+b}{3-b} &= -4 \\ 3+b &= -4(3-b) \\ 3+b &= -12+4b \\ 15 &= 3b \end{aligned}$$

and so $b = 5$.

Answer: (A)

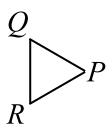
10. *Solution 1*
Since x is 20% of y, then $x = \frac{20}{100}y = \frac{1}{5}y$.
Since x is 50% of z, then $x = \frac{1}{2}z$.
Therefore, $\frac{1}{5}y = \frac{1}{2}z$ which gives $\frac{2}{5}y = z$.
Thus, $z = \frac{40}{100}y$ and so z is 40% of y.

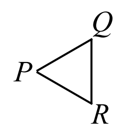
Solution 2
Since x is 20% of y, then $x = 0.2y$.
Since x is 50% of z, then $x = 0.5z$.
Therefore, $0.2y = 0.5z$ which gives $0.4y = z$.
Thus, $z = 0.4y$ and so z is 40% of y.

Answer: (D)

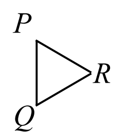
11. The store sells 250 g of jellybeans for \$7.50, which is 750 cents.
Therefore, 1 g of jellybeans costs $750 \div 250 = 3$ cents.
This means that \$1.80, which is 180 cents, will buy $180 \div 3 = 60$ g of jellybeans.

Answer: (C)

12. Starting with 

and flipping across QR results in 

.

Now flipping across PR results in 

, which is the resulting position that Paola sees.

Answer: (E)

13. Since $2 \times 2 \times 2 = 8$, a cube with edge length 2 has volume 8. (We note also that $\sqrt[3]{8} = 2$.)
Therefore, each of the cubes with volume 8 have a height of 2.
This means that the larger cube has a height of $2 + 2 = 4$, which means that its volume is $4^3 = 4 \times 4 \times 4 = 64$.

Answer: (E)

14. Since 100 000 does not include the block of digits 178, each integer between 10 000 and 100 000 that includes the block of digits 178 has five digits.
Such an integer can be of the form $178xy$ or of the form $x178y$ or of the form $xy178$ for some digits x and y.
The leading digit of a five-digit integer has 9 possible values (any digit from 1 to 9, inclusive) while a later digit in a five-digit integer has 10 possible vlaues (0 or any digit from 1 to 9, inclusive).
This means that

- there are 100 integers of the form $178xy$ (10 choices for each of x and y, and $10 \times 10 = 100$),
- there are 90 integers of the form $x178y$ (9 choices for x and 10 choices for y, and $9 \times 10 = 90$), and
- there are 90 integers of the form $xy178$ (9 choices for x and 10 choices for y, and $9 \times 10 = 90$).

In total, there are thus $100 + 90 + 90 = 280$ integers between 10 000 and 100 000 that include the block of digits 178.

Answer: (A)

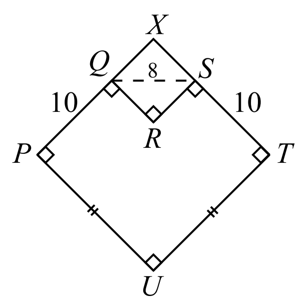
15. Since $a + 5 = b$, then $a = b - 5$.
Since $a = b - 5$ and $c = 5 + b$ and $b + c = a$, then

$$\begin{aligned} b + (5 + b) &= b - 5 \\ 2b + 5 &= b - 5 \\ b &= -10 \end{aligned}$$

(If $b = -10$, then $a = b - 5 = -15$ and $c = 5 + b = -5$ and $b + c = (-10) + (-5) = (-15) = a$, as required.)

Answer: (C)

16. Extend PQ and TS to meet at point X.



Since quadrilateral $QRSX$ has three right angles (at Q, R and S), it must have a fourth right angle at X.
Thus, $QRSX$ is a rectangle, which means that $XS = QR$ and $QX = RS$.
The perimeter of $PQRSTU$ is

$$\begin{aligned} PQ + QR + RS + ST + TU + UP &= PQ + XS + QX + ST + TU + UP \\ &= (PQ + QX) + (XS + ST) + TU + UP \\ &= PX + XT + TU + UP \end{aligned}$$

which is the perimeter of quadrilateral of $PXTU$.
But quadrilateral $PXTU$ has four right angles, and so is a rectangle.
Also, $PU = UT$, so $PXTU$ is a square, and so the perimeter of $PXTU$ equals

$$4 \times PX = 4 \times (PQ + QX) = 4 \times (10 + QX) = 40 + 4 \times QX$$

Finally, $QX = PX - PQ = PX - 10 = XT - 10 = XT - ST = XS$, which means that $\triangle QXS$ is isosceles as well as being right-angled at X.
By the Pythagorean Theorem, $QX^2 + XS^2 = QS^2$ and so $2 \times QX^2 = 8^2$ or $QX^2 = 32$.
Since $QX > 0$, then $QX = \sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$.
Thus, the perimeter of $PQRSTU$ is $40 + 4 \times 4\sqrt{2} = 40 + 16\sqrt{2} \approx 62.6$.
(We could have left this as $40 + 4\sqrt{32} \approx 62.6$.)
Of the given choices, this is closest to 63.

Answer: (C)

17. Zebadiah must remove at least 3 shirts.
If he removes 3 shirts, he might remove 2 red shirts and 1 blue shirt.
If he removes 4 shirts, he might remove 2 red shirts and 2 blue shirts.
Therefore, if he removes fewer than 5 shirts, it is not guaranteed that he removes either 3 of the same colour or 3 of different colours.
Suppose that he removes 5 shirts.
If 3 are of the same colour, the requirements are satisfied.
If no 3 of the 5 shirts are of the same colour, then at most 2 are of each colour. This means that he must remove shirts of 3 colours, since if he only removed shirts of 2 colours, he would remove at most $2 + 2 = 4$ shirts.
In other words, if he removes 5 shirts, it is guaranteed that there are either 3 of the same colours or shirts of all 3 colours.
Thus, the minimum number is 5.

ANSWER: (D)

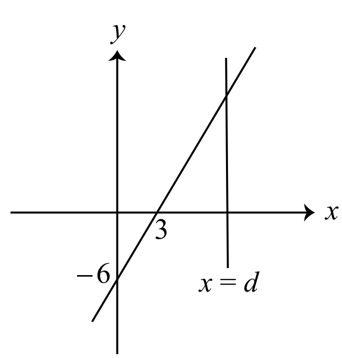
18. At the beginning of the first day, the box contains 1 black ball and 1 gold ball.
At the end of the first day, 2 black balls and 1 gold ball are added, so the box contains 3 black balls and 2 gold balls.
At the end of the second day, $2 \times 2 = 4$ black balls and $2 \times 1 = 2$ gold balls are added, so the box contains 7 black balls and 4 gold balls.
Continuing in this way, we find the following numbers of balls:

End of Day #	Black Balls	Gold Balls
2	7	4
3	$7 + 4 \times 2 = 15$	$4 + 4 = 8$
4	$15 + 8 \times 2 = 31$	$8 + 8 = 16$
5	$31 + 16 \times 2 = 63$	$16 + 16 = 32$
6	$63 + 32 \times 2 = 127$	$32 + 32 = 64$
7	$127 + 64 \times 2 = 255$	$64 + 64 = 128$

At the end of the 7th day, there are thus $255 + 128 = 383$ balls in the box.

ANSWER: (E)

19. The line with equation $y = 2x - 6$ has y-intercept -6.
Also, the x-intercept of $y = 2x - 6$ occurs when $y = 0$, which gives $0 = 2x - 6$ or $2x = 6$ which gives $x = 3$.
Therefore, the triangle bounded by the x-axis, the y-axis, and the line with equation $y = 2x - 6$ has base of length 3 and height of length 6, and so has area $\frac{1}{2} \times 3 \times 6 = 9$.
We want the area of the triangle bounded by the x-axis, the vertical line with equation $x = d$, and the line with equation $y = 2x - 6$ to be 4 times this area, or 36.
This means that $x = d$ is to the right of the point $(3, 0)$, because the new area is larger. In other words, $d > 3$.



The base of this triangle has length $d-3$, and its height is $2d-6$, since the height is measured along the vertical line with equation $x=d$.
Thus, we want $\frac{1}{2}(d-3)(2d-6)=36$ or $(d-3)(d-3)=36$ which means $(d-3)^2=36$.
Since $d-3>0$, then $d-3=6$ which gives $d=9$.
Alternatively, we could note that if similar triangles have areas in the ratio $4:1$ then their corresponding lengths are in the ratio $\sqrt{4}:1$ or $2:1$.
Since the two triangles in question are similar (both are right-angled and they have equal angles at the point $(3,0)$), the larger triangle has base of length $2\times 3=6$ and so $d=3+6=9$.

ANSWER: (A)

20. Since $3m^3$ is a multiple of 3, then $5n^5$ is a multiple of 3.
Since 5 is not a multiple of 3 and 3 is a prime number, then n^5 is a multiple of 3.
Since n^5 is a multiple of 3 and 3 is a prime number, then n is a multiple of 3, which means that $5n^5$ includes at least 5 factors of 3.
Since $5n^5$ includes at least 5 factors of 3, then $3m^3$ includes at least 5 factors of 3, which means that m^3 is a multiple of 3, which means that m is a multiple of 3.
Using a similar analysis, both m and n must be multiples of 5.
Therefore, we can write $m=3^a5^bs$ for some positive integers a, b and s and we can write $n=3^c5^dt$ for some positive integers c, d and t, where neither s nor t is a multiple of 3 or 5. (In other words, we have grouped all of the factors of 3 and 5 in each of m and n.)
From the given equation,

$$\begin{aligned}
3m^3 &= 5n^5 \\
3(3^a5^bs)^3 &= 5(3^c5^dt)^5 \\
3\times 3^{3a}5^{3b}s^3 &= 5\times 3^{5c}5^{5d}t^5 \\
3^{3a+1}5^{3b}s^3 &= 3^{5c}5^{5d+1}t^5
\end{aligned}$$

Since s and t are not multiples of 3 or 5, we must have $3^{3a+1}=3^{5c}$ and $5^{3b}=5^{5d+1}$ and $s^3=t^5$.
Since s and t are positive and m and n are to be as small as possible, we can set $s=t=1$, which satisfy $s^3=t^5$.
Since $3^{3a+1}=3^{5c}$ and $5^{3b}=5^{5d+1}$, then $3a+1=5c$ and $3b=5d+1$.
Since m and n are to be as small as possible, we want to find the smallest positive integers a,b,c,d for which $3a+1=5c$ and $3b=5d+1$.
Neither $a=1$ nor $a=2$ gives a value for $3a+1$ that is a multiple of 5, but $a=3$ gives $c=2$.
Similarly, $b=1$ does not give a value of $3b$ that equals $5d+1$ for any positive integer d, but $b=2$ gives $d=1$.
Therefore, the smallest possible values of m and n are $m=3^35^2=675$ and $n=3^25^1=45$, which gives $m+n=720$.
(We can verify by substitution that $m=675$ and $n=45$ satisfy the equation $3m^3=5n^5$.)

ANSWER: (C)

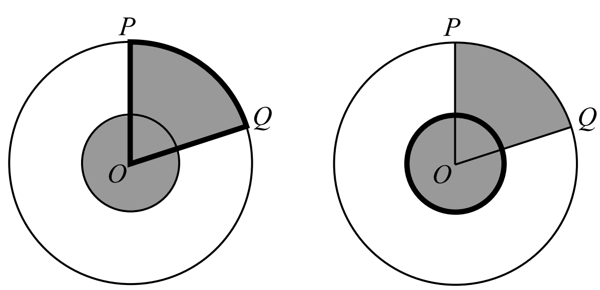
21. Since $20x + 11y = 881$, then $20x = 881 - 11y$ and $11y = 881 - 20x$.
Since x is an integer, then $20x$ is a multiple of 10 and so the units digit of $20x$ is 0 which means that the units digit of $881 - 20x$ is 1, and so the units digit of $11y$ is 1.
Since the units digit of $11y$ is 1, then the units digit of y is 1.
Since $20x$ is positive, then $11y = 881 - 20x$ is smaller than 881, which means that $11y < 881$ and so $y < \frac{881}{11} \approx 80.1$.
Thus, the possible values of y are 1, 11, 21, 31, 41, 51, 61, 71.
We check each of these:

y	$11y = 881 - 20x$	$20x$	x
1	11	870	Not an integer
11	121	760	38
21	231	650	Not an integer
31	341	540	27
41	451	430	Not an integer
51	561	320	16
61	671	210	Not an integer
71	781	100	5

Therefore, the sum of the smallest and largest of the permissible values of y is $11 + 71 = 82$.

ANSWER: 82

22. Since the shaded regions are equal, then when the unshaded sector in the small circle is shaded, the area of the now fully shaded sector of the larger circle must be equal to the area of the smaller circle.



The smaller circle has radius 1 and so has area $\pi \times 1^2 = \pi$.
The larger circle has radius 3 and so has area $\pi \times 3^2 = 9\pi$.
This means that the area of the shaded sector in the larger circle has area π, which means that it must be $\frac{1}{9}$ of the larger circle.
This means that $\angle POQ$ must be $\frac{1}{9}$ of a complete circle, and so $\angle POQ = \frac{1}{9} \times 360° = 40°$.
Thus, $x = 40$.

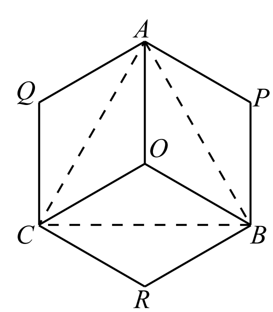
ANSWER: 40

23. Suppose that Andreas, Boyu, Callista, and Diane choose the numbers a, b, c, and d, respectively. There are 9 choices for each of a, b, c, and d, so the total number of quadruples (a, b, c, d) of choices is $9^4 = 6561$.
Among the 9 choices, 5 are odd $(1, 3, 5, 7, 9)$ and 4 are even $(2, 4, 6, 8)$.
If there are N quadruples (a, b, c, d) with $a + b + c + d$ even (that is, with the sum of their choices even), then the probability that the sum of their four integers is even is $\dfrac{N}{6561}$, which is in the desired form.
Therefore, we count the number of quadruples (a, b, c, d) with $a + b + c + d$ even.
Among the four integer a, b, c, d, either 0, 1, 2, 3, or 4 of these integers are even, with the remaining integers odd.
If 0 of a, b, c, d are even and 4 are odd, their sum is even.
If 1 of a, b, c, d is even and 3 are odd, their sum is odd.
If 2 of a, b, c, d are even and 2 are odd, their sum is even.
If 3 of a, b, c, d are even and 1 is odd, their sum is odd.
If 4 of a, b, c, d are even and 0 are odd, their sum is even.
Therefore, we need to count the number of quadruples (a, b, c, d) with 0, 2 or 4 even parts.

If 0 are even and 4 are odd, there are 5 choices for each of the parts, and so there are $5^4 = 625$ such quadruples.
If 4 are even and 0 are odd, there are 4 choices for each of the parts, and so there are $4^4 = 256$ such quadruples.
If 2 are even and 2 are odd, there are 4 choices for each of the even parts and 5 choices for each of the odd parts, and 6 pairs of locations for the even integers (ab, ac, ad, bc, bd, cd) with the odd integers put in the remaining two locations after the locations of the even integers are chosen. Thus, there are $4^2 \cdot 5^2 \cdot 6 = 2400$ such quadruples.
In total, this means that there are $625 + 256 + 2400 = 3281$ quadruples and so $N = 3281$.
The sum of the squares of the digits of N is equal to $3^2 + 2^2 + 8^2 + 1^2 = 9 + 4 + 64 + 1 = 78$.

ANSWER: 78

24. Let O be the vertex of the cube farthest away from the table.
Let A, B and C be the vertices of the cube connected to O with edges.
Since the cube has edge length 8, then $OA = OB = OC = 8$.
Note that $\angle AOB = \angle AOC = \angle BOC = 90^\circ$, which means that each of $\triangle AOB$, $\triangle AOC$ and $\triangle BOC$ is a right-angled isosceles triangle, which means that $AB = \sqrt{2}AO = 8\sqrt{2}$, and so $AB = AC = BC = 8\sqrt{2}$.
Let P, Q and R be the vertices that complete the square faces $PAOB$, $QAOC$ and $RBOC$.
From directly above, the cube looks like this:



When the sun is directly overhead, the shadow of the cube will look exactly like the area of the "flat" hexagon $APBRCQ$. In mathematical terms, we are determining the area of what is called a *projection*.

To find the area of this figure, we need to know some lengths, but we have to be careful because not all of the edges in the edges in the diagram above are "flat".

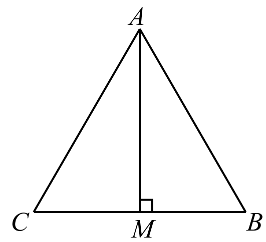
We do know that points A, B and C are in the same horizontal plane, and we also know that $AB = AC = BC = 8\sqrt{2}$. This means that these are true lengths in that the points A, B and C are this distance apart.

Note that flat quadrilateral $PAOB$ is divided into two regions of equal area by AB. Similarly, $QAOC$ is divided into two regions of equal area by AC, and $RBOC$ is divided into two regions of equal area by BC.

In other words, the area of $\triangle ABC$ is half of the area of hexagon $APBRCQ$, so to find the area of $APBRCQ$, we double the area of $\triangle ABC$.

So we need to calculate the area of equilateral $\triangle ABC$ which has side length $8\sqrt{2}$.

Let M be the midpoint of BC. Thus, $BM = CM = 4\sqrt{2}$.



Since $\triangle ABC$ is equilateral, then AM is perpendicular to BC.

By the Pythagorean Theorem in $\triangle AMC$,

$$AM = \sqrt{AC^2 - MC^2} = \sqrt{(8\sqrt{2})^2 - (4\sqrt{2})^2} = \sqrt{128 - 32} = \sqrt{96}$$

Therefore, the area of $\triangle ABC$ is

$$\tfrac{1}{2} \cdot CB \cdot AM = \tfrac{1}{2} \cdot 8\sqrt{2} \cdot \sqrt{96} = 4\sqrt{192} = 4\sqrt{64 \cdot 3} = 4 \cdot 8\sqrt{3} = 32\sqrt{3}$$

This means that the area of the hexagonal shadow is $64\sqrt{3}$.

Since this is in the form $a\sqrt{b}$ where a and b are positive integers and b is not divisible by the square of an integer larger than 1, then $a = 64$ and $b = 3$ and so $a + b = 67$.

Answer: 67

25. We begin by tracing what happens when $T = 337$.

We start with tokens labelled $1, 2, 3, \ldots, 335, 336, 337$ arranged around a circle.

We remove the first token (1), move 2 tokens along and remove that token (3), move 2 tokens along and remove that token (5), and continue around the circle until we remove tokens 335 and 337.

This leaves tokens $2, 4, 6, \ldots, 332, 334, 336$. These tokens differ by 2.

On the second pass, we start with tokens labelled $2, 4, 6, \ldots, 332, 334, 336$, which differ by 2.

Because the last token was removed on the first pass (337), the first token is not removed on the second pass, which means that we remove every other token starting with 4.

This means that the remaining tokens differ by 4, and are $2, 6, 10, \ldots, 326, 330, 334$.

On the third pass, we start with $2, 6, 10, \ldots, 326, 330, 334$, which differ by 4.

Because the last token (336) was removed on the previous pass, we remove every other token starting with 6.

The remaining tokens differ by 8 and are $2, 10, 18, \ldots, 314, 322, 330$.

On the fourth pass, we start with $2, 10, 18, \ldots, 314, 322, 330$, which differ by 8.
Because the last token (334) was removed on the previous pass, we remove every other token starting with 10.
The remaining tokens differ by 16 and are $2, 18, 34, \ldots, 290, 306, 322$.

On the fifth pass, we start with $2, 18, 34, \ldots, 290, 306, 322$, which differ by 16.
Because the last token (330) was removed on the previous pass, we remove every other token starting with 18.
The remaining tokens differ by 32 and are $2, 34, 66, 98, 130, 162, 194, 226, 258, 290, 322$.

On the sixth pass, we start with $2, 34, 66, 98, 130, 162, 194, 226, 258, 290, 322$, which differ by 32.
Because the second last token (306) was removed on the previous pass, we remove ever other token starting with 2.
This leaves $34, 98, 162, 226, 290$, which differ by 64.

On the seventh pass, we remove starting with the second token, which leaves $34, 162, 290$, which differ by 128.

On the eighth pass, we remove starting with the first token, which leaves 162.

This tells us that the smallest possible value of T is at least 162 and at most 337.

Next, we will show that $T = 209$ also gives a final token of 162 by working through the various passes. (This is a place where how we *discover* the answer is more than likely different than how we *justify* the answer.)
Before first pass:

$$1, 2, 3, \ldots, 207, 208, 209$$

After first pass (removing every other token starting with 1):

$$2, 4, 6, \ldots, 204, 206, 208$$

After second pass (removing every other token starting with 4):

$$2, 6, 10, \ldots, 198, 202, 206$$

After third pass (removing every other token starting with 6):

$$2, 10, 18, \ldots, 186, 194, 202$$

After fourth pass (removing every other token starting with 10):

$$2, 18, 34, \ldots, 162, 178, 194$$

After fifth pass (removing every other token starting with 18):

$$2, 34, 66, 98, 130, 162, 194$$

After sixth pass (removing every other token starting with 2):

$$34, 98, 162$$

After seventh pass (removing 98):

$$34, 162$$

This leaves 162 after the eighth pass, since 34 will be removed.
Therefore, when $T = 209$ and when $T = 337$, the final token is 162.

Finally, we show that if the final token starting with T tokens is 162, then $T \geq 209$, which will tell us that the smallest value of T is 209.
Suppose that $T \leq 209$ and that the token remaining after the final pass is 162.
Before each pass, the remaining tokens differ by a power of 2, since we start by removing every other token from a list that differs by 1, then every other token from a list that differs by 2, and so on.
The smallest powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256.

Since 162 is left after the last pass (this will turn out to be the eighth pass), the remaining tokens must have differed by 128 before the eighth pass, and thus were 34, 162. (Since $T \leq 209$, then there could not be a token numbered $162 + 128 = 290$.) Also, if the tokens differed by 64 before the eighth pass, there would have been tokens labelled 34 and 98 that were both removed.
Thus, before the eighth pass, the tokens were 34, 162 and 34 was removed.

Before the seventh pass, the tokens differed by 64.
Thus, these were 34, 98, 162. We note that the last token was not removed on this pass, and so the first token is removed on the eighth pass, as expected.
Also, there cannot be a token numbered $162 + 64 = 226$, since $T \leq 209$.

Before the sixth pass, the tokens differed by 32.
Thus, these were 2, 34, 66, 98, 130, 162, 194. The last token cannot have been 162 since the last token must be removed on this pass so that the second token (98) is removed on the seventh pass. Thus, $162 + 32 = 194$ must be the last token here. (Note that 226 was already rejected earlier.)

Before the fifth pass, the tokens differed by 16.
Since the first token (2) is removed on the sixth pass, the last token is not removed on the fifth pass.
This means that the tokens before this pass were

$$2, 18, 34, 50, 66, 82, 98, 114, 130, 146, 162, 178, 194$$

On this pass, 178 is the last token removed. (Note that $194 + 16 = 210$ is too large.)

Before the fourth pass, the tokens differed by 8.
Since the second token (18) is removed on the fifth pass, the last token is removed on the fourth pass.
This means that the tokens before this pass were

$$2, 10, 18, \ldots, 162, 170, 178, 186, 194, 202$$

The token 202 must be included since 194 remains for the fifth pass. (Note that $202 + 8 = 210$ is too large.)

Before the third pass, the tokens differed by 4.
Since the second token (10) is removed on the fourth pass, the last token is removed on the third pass.
This means that the tokens before this pass were

$$2, 6, 10, 14, 18, \ldots, 186, 190, 194, 198, 202, 206$$

The token 206 must be included since 202 remains for the fourth pass. (Note that $206 + 4 = 210$ is too large.)

Before the second pass, the tokens differed by 2.
Since the second token (6) is removed on the third pass, the last token is removed on the second pass.
This means that the tokens before this pass were

$$2, 4, 6, 8, \ldots, 198, 200, 202, 204, 206, 208$$

The token 208 must be included since 206 remains for the third pass. (Note that $208 + 2 = 210$ is too large.)

Before the first pass, the tokens differed by 1.
Since the second token (4) is removed on the second pass, the last token is removed on the first pass.
This means that the tokens before this pass were

$$1, 2, 3, 4, \ldots, 204, 205, 206, 207, 208, 209$$

The token 209 must be included since 208 remains for the second pass. (Note that $209+1 = 210$ is too large.)

Therefore, we must have at least 209 tokens for the final token to be 162, and so the smallest possible value of T is 209, whose rightmost two digits are 09.

ANSWER: 09