The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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# *2022 Canadian Intermediate Mathematics Contest*

**Wednesday, November 16, 2022**
**(in North America and South America)**

**Thursday, November 17, 2022**
**(outside of North America and South America)**

*Solutions*

**Part A**

1. Since the area of one square face of the cube is 16 $\text{cm}^2$, the edge length of the cube is $\sqrt{16 \text{ cm}^2}$ which equals 4 cm.
   Since the edge length of the cube is 4 cm, its volume is $(4 \text{ cm})^3 = 4^3 \text{ cm}^3 = 64 \text{ cm}^3$.
   Thus, $V = 64$.

   ANSWER: 64

2. Since $\triangle ADC$ is isosceles with $AD = DC$, we have $\angle DAC = \angle DCA = 40^\circ$.
   Since the sum of the measures of the angles in a triangle is $180^\circ$, we have

   $$\angle ADC = 180^\circ - \angle DAC - \angle DCA = 180^\circ - 40^\circ - 40^\circ = 100^\circ$$

   Since a straight angle measures $180^\circ$, we have $\angle ADB = 180^\circ - \angle ADC = 180^\circ - 100^\circ = 80^\circ$.
   Since $\triangle ADB$ is isosceles with $AD = DB$, we have $\angle DAB = \angle DBA$ and so

   $$\angle DAB = \tfrac{1}{2}(180^\circ - \angle ADB) = \tfrac{1}{2}(180^\circ - 80^\circ) = 50^\circ$$

   Finally, $\angle BAC = \angle DAB + \angle DAC = 50^\circ + 40^\circ = 90^\circ$.

   We note that students who know about "exterior angles" could shorten this solution by noting that, for example, $\angle ADB = \angle DAC + \angle DCA = 40^\circ + 40^\circ = 80^\circ$.
   Additionally, some students may note that since $DA = DB = DC$, then $D$ is the centre of the circle that passes through $A$, $B$ and $C$ (that is, the circumcircle of $\triangle ABC$). Since $BC$ will be a diameter of this circle with $A$ a point on its circumference, then $\angle BAC = 90^\circ$.

   ANSWER: $90^\circ$

3. Since Marie-Pascale solves 4 math problems per day and solves 72 problems in total, this takes her $72 \div 4 = 18$ days.
   Since Kaeli solves 54 more problems than Marie-Pascale, she solves $72 + 54 = 126$ problems.
   Since Kaeli solves $x$ problems per day over 18 days and solves 126 problems in total, then $x = \dfrac{126}{18} = 7$.

   ANSWER: 7

4. *Solution 1*
   Since $\dfrac{c}{d} = \dfrac{7}{15}$, then $c = \dfrac{7}{15}d$.
   Since $\dfrac{c}{b} = \dfrac{1}{5}$, then $b = 5c = 5 \times \dfrac{7}{15}d = \dfrac{7}{3}d$.
   Since $\dfrac{a}{b} = \dfrac{2}{3}$, then $a = \dfrac{2}{3}b = \dfrac{2}{3} \times \dfrac{7}{3}d = \dfrac{14}{9}d$.
   Thus,

   $$\frac{ab}{cd} = \frac{(14/9)d \times (7/3)d}{(7/15)d \times d} = \frac{(98/27)d^2}{(7/15)d^2} = \frac{98}{27} \times \frac{15}{7} = \frac{14}{9} \times \frac{5}{1} = \frac{70}{9}$$

*Solution 2*

Suppose that $d = 45$.

Since $\dfrac{c}{d} = \dfrac{7}{15}$, then $c = \dfrac{7}{15}d = \dfrac{7}{15} \times 45 = 21$.

Since $\dfrac{c}{b} = \dfrac{1}{5}$, then $b = 5c = 5 \times 21 = 105$.

Since $\dfrac{a}{b} = \dfrac{2}{3}$, then $a = \dfrac{2}{3}b = \dfrac{2}{3} \times 105 = 70$.

Therefore, $\dfrac{ab}{cd} = \dfrac{70 \times 105}{21 \times 45} = \dfrac{10 \times 7}{3 \times 3} = \dfrac{70}{9}$.

This solution does not show that this is the value of $\dfrac{ab}{cd}$ for all such values of $a$, $b$, $c$, and $d$, but the text of the problem implies that the value is the same for any choice of $a$, $b$, $c$, and $d$ that satisfy the restrictions. Solution 1 justifies that this is the value for all choices of $a$, $b$, $c$, and $d$.

Answer: $\dfrac{70}{9}$

5. When three numbers $a$, $b$, $c$ have the property that the middle number, $b$, is the average of the other two, then $b = \dfrac{a+c}{2}$ tells us that $2b = a + c$ or $b + b = a + c$ and so $b - c = a - b$.
In other words, if $b - c = d$ (that is, if $b = c + d$ for some number $d$), we have $a - b = d$, which gives $a = b + d = (c + d) + d = c + 2d$.
We note that the common difference $d$ could be positive or negative.
Since Viswanathan's three scores have this property and his score in the third game is 25, then his scores in the first two games can be written as $25 + 2d$ and $25 + d$, as shown in the table:

| | 1st game | 2nd game | 3rd game |
|---|---|---|---|
| Viswanathan | $25 + 2d$ | $25 + d$ | 25 |
| Magnus | | | |

Since Viswanathan wins either the first game or the second game, his score in this game must be at least 25. This means that $d \geq 0$; since the scores are all distinct, it must be the case that $d > 0$.
Since Viswanathan's scores in these games are both larger than 25, Magnus's scores in these games are either 2 more and 2 less than Viswanathan's or 2 less and 2 more than Viswanathan's, depending on the order in which they won these games.

Suppose that Viswanathan wins the first game and Magnus wins the second game.
Then Magnus's score in the first game is 2 less than Viswanathan's (and so is $23 + 2d$) and Magnus's score in the second game is 2 more than Viswanathan's (and so is $27 + d$).

| | 1st game | 2nd game | 3rd game |
|---|---|---|---|
| Viswanathan | $25 + 2d$ | $25 + d$ | 25 |
| Magnus | $23 + 2d$ | $27 + d$ | |

Since scores of $a$, $b$, $c$ in order of the three games gives $2b = a + c$, then $c = 2b - a$.
Here, this means that Magnus's score in the third game is $2(27 + d) - (23 + 2d) = 31$, which is not possible since a final score of 31 to 25 is not possible.

Suppose that Magnus wins the first game and Viswanathan wins the second game.
Then Magnus's score in the first game is 2 more than Viswanathan's (and so is $27 + 2d$) and Magnus's score in the second game is 2 less than Viswanathan's (and so is $23 + d$).

| | 1st game | 2nd game | 3rd game |
|---|---|---|---|
| Viswanathan | $25+2d$ | $25+d$ | 25 |
| Magnus | $27+2d$ | $23+d$ | |

Here, Magnus's score in the third game is $2(23+d)-(27+2d)=19$, which is possible since a score of 25 to 19 is possible.
Therefore, Magnus's score in the third game is 19. We note that this answer does not depend on $d$, which tells us that this score will be 19 regardless of the value of $d>0$.

ANSWER: 19

6. For the product $abc$ to be a multiple of 12, the integers $a$, $b$ and $c$ must include between them at least 1 factor of 3 and 2 factors of 2.
Among the integers in the list, there are five categories with respect to divisibility by 2 and 3:

- (A): 3 are not divisible by 2 or 3 (these are 1, 5, 37)
- (B): 3 are divisible by 2 but not by 4 or 3 (these are 22, 46, 50)
- (C): 1 is divisible by 4 but not by 3 (this is 8)
- (D): 4 are divisible by 3 but not by 2 (these are 21, 27, 33, 39)
- (E): 1 is divisible by 2 and by 3 (this is 30)

We work through two possibilities: either 30 is chosen or 30 is not chosen. We do not need to worry about whether 30 is $a$, $b$ or $c$, since the three chosen integers can be arranged in increasing order after they are chosen.

<u>Case 1: 30 is chosen</u>
In this case, at least one of the remaining integers is even, since 30 includes only 1 factor of 2.

If both of the remaining integers are even, then 2 of the 4 integers from (B) and (C) are chosen. There are 6 ways to do this: 22 and 46; 22 and 50; 22 and 8; 46 and 50; 46 and 8; 50 and 8. Therefore, there are 6 ways to choose in this sub-case.

If only one of the remaining integers is even, this even integer can come from either (B) or (C). There are 4 such integers.
The third integer chosen must be odd, and so is from (A) or (D). There is a total of 7 integers in these.
Thus, in this sub-case there are $4\times 7=28$ ways to choose since each of the 4 even integers can be paired with each of the 7 odd integers.

<u>Case 2: 30 is not chosen</u>
In this case, one or two of the three integers chosen must be from (D), since $abc$ must include at least 1 factor of 3. (If all three were from (D), there would be no factors of 2.)

If two of the integers are from (D), there are 6 ways of choosing these 2 integers (21 and 27; 21 and 33; 21 and 39; 27 and 33; 27 and 39; 33 and 39).
The third integer chosen must then be a multiple of 4 in order for $abc$ to have at least 2 factors of 2. Thus, the third integer is 8.
Thus, in this sub-case there are 6 ways to choose.

If one of the integers is from (D), there are 4 ways of choosing that integer.
The remaining two integers chosen then need to include a multiple of 4 and another even integer, or a multiple of 4 and an odd integer not divisible by 3, or two even integers that are

not multiples of 4. (Can you see why these are all of the possible cases?)
If these integers are a multiple of 4 and another even integer, the multiple of 4 must be 8 and the even integer comes from the 3 integers in (B).
Thus, there are $4 \times 3 = 12$ ways to choose in this sub-case, since each of the 4 integers from (D) can be paired with each of the 3 integers from (B).
If the two integers are a multiple of 4 and an odd integer not divisible by 3, the multiple of 4 must be 8 and the odd integer comes from the 3 integers in (A).
Thus, there are $4 \times 3 = 12$ ways to choose in this sub-case.
If the two integers are even and not multiples of 4, we choose 2 of the 3 integers from (B) and there are 3 ways to do this, as we saw earlier.
Thus, there are $4 \times 3 = 12$ ways to choose in this sub-case.

In total, there are $6 + 28 + 6 + 12 + 12 + 12 = 76$ ways to choose the three integers.

Answer: 76

## Part B

1. (a) Since the sum of the entries in the left column is 29, the bottom left entry is $29-13-9=7$. Since the sum of the entries in the horizontal row is 29, the remaining two entries have a sum of $29-9=20$. Since the possible entries are 3, 5 and 15, these entries must be 5 and 15.
This means that the 3 must go in the bottom right box, which means that the middle box in the right column must contain $29-11-3=15$, which means that 5 goes in the middle box.

| | | |
|---|---|---|
| 13 | | 11 |
| 9 | 5 | 15 |
| 7 | | 3 |

(b) The sums of the left and right columns are equal. Thus,

$$15+(t+1)+11=(2t-3)+10+14$$
$$t+27=2t+21$$
$$6=t$$

If $t=6$, we complete the H to verify that the entries are all different:

| | | |
|---|---|---|
| 15 | | 9 |
| 7 | 16 | 10 |
| 11 | | 14 |

(c) Since the three sums are equal, we have $a+12+c=12+9+7=b+7+11$ and so $a+c+12=28=b+18$ which gives $a+c=16$ and $b=10$.
Since $a+c=16$ and $a$ and $c$ are positive integers with $a<c$, we have the following possibilities:

- $a=1$ and $c=15$; all 7 integers are different in this case
- $a=2$ and $c=14$; all 7 integers are different in this case
- $a=3$ and $c=13$; all 7 integers are different in this case
- $a=4$ and $c=12$; this is not possible since there would be two 12s
- $a=5$ and $c=11$; this is not possible since there would be two 11s
- $a=6$ and $c=10$; this is not possible since there would be two 10s
- $a=7$ and $c=9$; this is not possible since there would be two 7s (and two 9s)

Thus, the possible values of $a$ are 1, 2 and 3, giving the following figures:

| | | |
|---|---|---|
| 1 | | 10 |
| 12 | 9 | 7 |
| 15 | | 11 |

| | | |
|---|---|---|
| 2 | | 10 |
| 12 | 9 | 7 |
| 14 | | 11 |

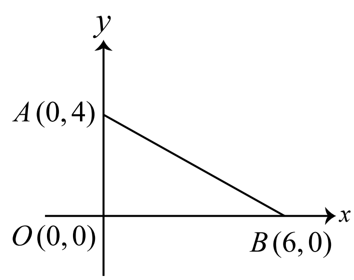
| | | |
|---|---|---|
| 3 | | 10 |
| 12 | 9 | 7 |
| 13 | | 11 |

(d) Since the three sums are equal, we have $7+10+k = 10+n+18 = (n+6)+18+4$ which gives $k+17 = n+28 = n+28$, which means that $k = n+11$.
Since $k$ is between 4 and 18, inclusive, and $k = n+11$, then we must have $n \leq 7$. Since $n$ is between 4 and 18, inclusive, then $n \geq 4$ as well.
Putting this together, $n$ can equal 4, 5, 6, or 7, which give values for $k$ of 15, 16, 17, or 18, respectively, and values of $n+6$ of 10, 11, 12, or 13, respectively.
Since the figure already contains 10 and 18, we cannot have $k = 18$ or $n+6 = 10$, which means that $n \neq 7$ and $n \neq 4$.
This means that the possible values of $k$ are 16 and 17, which gives the following figures:

| | | |
|---|---|---|
| 7 | | 11 |
| 10 | 5 | 18 |
| 16 | | 4 |

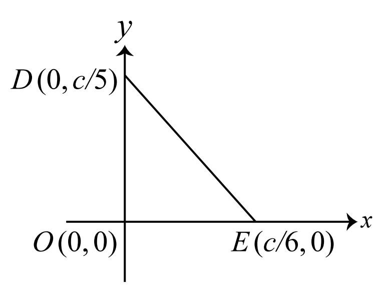
| | | |
|---|---|---|
| 7 | | 12 |
| 10 | 6 | 18 |
| 17 | | 4 |

2. (a) When $x = 0$, the equation $2x + 3y = 12$ becomes $3y = 12$ and so $y = 4$. Thus, the coordinates of $A$ are $(0, 4)$.
When $y = 0$, the equation $2x + 3y = 12$ becomes $2x = 12$ and so $x = 6$. Thus, the coordinates of $B$ are $(6, 0)$.
Thus, the vertices of $\triangle AOB$ are $A(0, 4)$, $O(0, 0)$ and $B(6, 0)$.



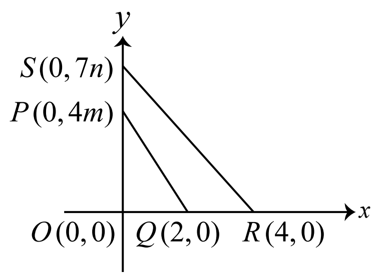

Since $\triangle AOB$ is right-angled at $O$, its area equals $\frac{1}{2} \times AO \times BO = \frac{1}{2} \times 4 \times 6 = 12$.

(b) When $x = 0$, the equation $6x + 5y = c$ becomes $5y = c$ and so $y = \frac{1}{5}c$. Thus, the coordinates of $D$ are $(0, \frac{1}{5}c)$.
When $y = 0$, the equation $6x + 5y = c$ becomes $6x = c$ and so $x = \frac{1}{6}c$. Thus, the coordinates of $E$ are $(\frac{1}{6}c, 0)$.
Thus, the vertices of $\triangle DOE$ are $D(0, \frac{1}{5}c)$, $O(0, 0)$ and $E(\frac{1}{6}c, 0)$.




Since $\triangle DOE$ is right-angled at $O$, its area equals $\frac{1}{2} \times DO \times EO = \frac{1}{2} \times \frac{1}{5}c \times \frac{1}{6}c$.
Since the area of $\triangle DOE$ is equal to 240, we obtain $\frac{1}{2} \times \frac{1}{5}c \times \frac{1}{6}c = 240$, which gives $\frac{1}{60}c^2 = 240$.
Thus, $c^2 = 60 \times 240 = 14\,400$. Since $c > 0$, then $c = 120$.

(c) When $x = 0$, the equation $(2m)x + y = 4m$ becomes $y = 4m$. Thus, the coordinates of $P$ are $(0, 4m)$.
When $y = 0$, the equation $(2m)x + y = 4m$ becomes $(2m)x = 4m$ and so $x = 2$. (We note that $100 \leq m < n$ and so neither $m$ nor $n$ can equal 0.) Thus, the coordinates of $Q$ are $(2, 0)$.
When $x = 0$, the equation $(7n)x + 4y = 28n$ becomes $4y = 28n$ and so $y = 7n$. Thus, the coordinates of $S$ are $(0, 7n)$.
When $y = 0$, the equation $(7n)x + 4y = 28n$ becomes $(7n)x = 28n$ and so $x = 4$. Thus, the coordinates of $R$ are $(4, 0)$.
Since $n > m$ and $7 > 4$, then $7n > 4m$ and so $S$ is farther up the $y$-axis than $P$.




To calculate the area of quadrilateral $PQRS$, we subtract the area of $\triangle POQ$ from the area of $\triangle SOR$.
Therefore,

$$\begin{aligned}
2022 &= \tfrac{1}{2} \times SO \times RO - \tfrac{1}{2} \times PO \times QO \\
2022 &= \tfrac{1}{2} \times 7n \times 4 - \tfrac{1}{2} \times 4m \times 2 \\
2022 &= 14n - 4m \\
1011 &= 7n - 2m
\end{aligned}$$

We want to find two pairs $(m, n)$ of positive integers with $100 \leq m < n$ and for which $7n - 2m = 1011$.
When $m = 100$, we get $7n - 200 = 1011$ and so $7n = 1211$ which gives $n = 173$. The pair $(m, n) = (100, 173)$ satisfies the requirements.
From $7 \times 173 - 2 \times 100 = 1011$, we add and subtract 14 to obtain

$$7 \times 173 + 14 - 2 \times 100 - 14 = 1011$$

which gives

$$7 \times 173 + 7 \times 2 - 2 \times 100 - 2 \times 7 = 1011$$

and so

$$7 \times 175 - 2 \times 107 = 1011$$

Therefore, $(m, n) = (107, 175)$ is a second pair that satisfies the given requirements.

3. (a) *Solution 1*

Since 10 minutes is $\frac{1}{6}$ of an hour, when Beatrice walks at 5 km/h for 10 minutes, she walks a distance of 5 km/h $\times \frac{1}{6}$ h $= \frac{5}{6}$ km.

Since Hieu cycles at 15 km/h and Beatrice walks at 5 km/h, then Hieu catches up to Beatrice at a rate of 15 km/h $-$ 5 km/h $=$ 10 km/h.

In other words, Hieu closes the $\frac{5}{6}$ km gap at a rate of 10 km/h, which takes $\dfrac{\frac{5}{6}\text{ km}}{10\text{ km/h}}$, or $\dfrac{5}{60}$ h, or 5 minutes.

*Solution 2*

When Beatrice walks at 5 km/h for 10 minutes, she walks a distance of 5 km/h $\times \frac{1}{6}$ h or $\frac{5}{6}$ km.
Suppose that Hieu takes $t$ hours to catch up to Beatrice.
Since Hieu cycles at 15 km/h, Hieu cycles 15 km/h $\times\, t$ h $= 15t$ km in total over the course of $t$ hours.
Using similar reasoning, Beatrice walks $5t$ km after her initial $\frac{5}{6}$ km.
When Hieu catches up to Beatrice, they have travelled exactly the same distance, and so $15t = 5t + \frac{5}{6}$ which gives $10t = \frac{5}{6}$ and so $t = \frac{5}{60}$.
Thus, Hieu catches up to Beatrice after $\frac{5}{60}$ h, which is equivalent to 5 minutes.

(b) (i) There are 60 integers in the interval from 0 to 59 inclusive.
This means that there are 60 possible values for $b$ and 60 possible values for $h$.
Since $b$ and $h$ are chosen randomly and independently, then there are $60 \times 60 = 3600$ possible pairs of values for $b$ and $h$.
We say that Beatrice and Hieu meet if they are at the same place on the path at the same time, including possibly starting or ending together.
To determine the probability that Beatrice and Hieu meet, we count the number of pairs of values for $b$ and $h$ for which they meet, and then divide this total by 3600.

Since the path is 2 km long and Beatrice walks at 5 km/h, then Beatrice takes $\dfrac{2\text{ km}}{5\text{ km/h}}$ or $\dfrac{2}{5}$ h or 24 minutes to walk the length of the path.

Since the path is 2 km long and Hieu cycles at 15 km/h, then Hieu takes $\dfrac{2\text{ km}}{15\text{ km/h}}$ or $\dfrac{2}{15}$ h or 8 minutes to cycle the length of the path.

Since Beatrice walks the path in 24 minutes and Hieu cycles the path in 8 minutes, then Beatrice and Hieu will meet on the path whenever $h - b$ is at least 0 and at most $24 - 8 = 16$. Here is an explanation for why this is true:

- If Hieu starts before Beatrice (that is, if $h < b$), since Hieu is faster than Beatrice, then Beatrice cannot catch up to Hieu, and so they never meet on the path.
- If Hieu starts at the same time as Beatrice (that is, if $h = b$), then they meet at the start of the path.
- Suppose that Hieu starts after Beatrice (that is, if $h > b$) but fewer than 16 minutes after Beatrice (that is, $h - b < 16$).
  We note that Hieu finishes at $h + 8$ minutes after 9:00 a.m., and Beatrice finishes at $b + 24$ minutes after 9:00 a.m.
  Since $h - b < 16$, then $h < b + 16$ and so $h + 8 < b + 24$, which means that Hieu

finishes before Beatrice.
Since Hieu starts after Beatrice and finishes before Beatrice, then Hieu must pass Beatrice along the path, and so they meet.

- If Hieu starts exactly 16 minutes after Beatrice (that is, if $h - b = 16$), then Hieu and Beatrice finish at exactly the same time (because $h+8 = b+24$) and so meet at the end of the path.
- If Hieu starts more than 16 minutes after Beatrice (that is, if $h - b > 16$), then Hieu will not catch up to Beatrice. In this case, this is because Hieu is "behind" where Hieu would be if Hieu started exactly 16 minutes after Beatrice, and in the previous case, Hieu just caught up to Beatrice at the very end of the path.

So we need to count the number of pairs of integer values for $h$ and $b$ with $h$ and $b$ between 0 and 59, inclusive, and with $h - b \geq 0$ and $h - b \leq 16$.
To do this, we work through the values of $b$ from 0 to 59, in each case determining the number of values of $h$ that satisfy the restrictions.
If $b = 0$, then $h$ can be any of 0 to 16, inclusive. There are 17 such values.
If $b = 1$, then $h$ can be any of 1 to 17, inclusive. There are again 17 such values.
In general, if $b \geq 0$ and $b \leq 43$, then $h$ can be any of $b$ to $b + 16$ inclusive, and since $b \leq 43$, we have $b + 16 \leq 59$, which means that each of these 17 values satisfy the required conditions. Note that there are 44 values of $b$ for which there are 17 values of $h$.
If $b = 44$, then $h$ can be any of 44 to 59, inclusive. We note while $h = 60$ is 16 greater than $b = 44$, it is not an allowed value for $h$. In this case, there are 16 values for $h$.
If $b = 45$, then $h$ can be any of 45 to 59, inclusive. There are 15 values for $h$.
As $b$ increases by 1 from 45 to 59, the number of values of $h$ decreases by 1 in each case, since the maximum value of $h$ does not change while its minimum value increases by 1.
When $b = 59$, there is 1 value of $h$ (namely, $h = 59$) that satisfies the conditions.
Therefore, if $N$ is the number of pairs of values of $h$ and $b$ that satisfy the given conditions, then

$$\begin{aligned}
N &= 44 \times 17 + 16 + 15 + 14 + \cdots + 3 + 2 + 1 \\
&= 44 \times 17 + (16 + 1) + (15 + 2) + \cdots + (10 + 7) + (9 + 8) \\
&= 44 \times 17 + 8 \times 17 \\
&= 52 \times 17 \\
&= 884
\end{aligned}$$

Finally, this means that the probability that Hieu and Beatrice are at the same place on the path at the same time is $\dfrac{N}{3600}$ or $\dfrac{884}{3600}$, which simplifies to $\dfrac{221}{900}$.

(ii) As in (i), there are 3600 possible pairs of values for $b$ and $h$.
If $M$ of these pairs result in Hieu and Beatrice meeting, then the probability that they meet is $\dfrac{M}{3600}$.
We are told that the probability is $\dfrac{13}{200}$, and so $\dfrac{M}{3600} = \dfrac{13}{200}$ which means that $M$ is equal to $\dfrac{13 \times 3600}{200} = 13 \times 18 = 234$.
As in (i), Hieu takes 8 minutes to cycle the length of the path.

Since Beatrice rides a scooter at $x$ km/h (with $x > 5$ and $x < 15$), then it takes Beatrice $\dfrac{2\text{ km}}{x\text{ km/h}} = \dfrac{2}{x}$ h $= \dfrac{120}{x}$ min to travel the length of the path.

Since $x > 5$, then $\dfrac{120}{x} < 24$, and so Hieu is still faster than Beatrice.

Since $x < 15$, then $\dfrac{120}{x} > 8$.

Using a similar analysis to that in (i), Hieu and Beatrice meet whenever $h - b$ is at least 0 and at most $\dfrac{120}{x} - 8$.

This means that we need to determine the range of possible values of $x$ for which there are 234 pairs of integer values of $h$ and $b$ for which each of $h$ and $b$ is between 0 and 59, inclusive, and $h - b \geq 0$ and $h - b \leq \dfrac{120}{x} - 8$.

Since $h$ and $b$ are integers, then $h - b$ is an integer.

Let $t$ be the largest integer less than or equal to $\dfrac{120}{x} - 8$.

Note that, since $t$ is the largest such integer, $t + 1$ is greater than $\dfrac{120}{x} - 8$.

Note also that $\dfrac{120}{x} - 8$ is greater than 0 and less than 16, which means that $t$ is between 0 and 15, inclusive. This means that subject to the range restrictions on $h$ and $b$, the value of $h - b$ can be from 0 to $t$, inclusive.

As in (i), we count the number of pairs of values for $h$ and $b$ by working systematically through the values of $b$.

When $b = 0$, there are $t + 1$ possible values for $h$, namely the integers from 0 to $t$, inclusive.

When $b = 1$, there are $t + 1$ possible values for $h$, namely the integers from 1 to $t + 1$, inclusive.

Similarly, while $b \leq 59 - t$, there are $t + 1$ possible values for $h$. Since these values of $b$ run from 0 to $59 - t$, inclusive, there are $60 - t$ such values of $b$.

When $b = (59 - t) + 1 = 60 - t$, the largest possible value of $h$ is still 59 (which corresponds to $h - b = t - 1$) and so there are $t$ possible values for $h$.

As in (i), the number of possible values of $h$ decreases by 1 each time $b$ increases by 1 until $b = 59$.

Since the total number of pairs of values for $b$ and $h$ is 234, then

$$\begin{aligned}
234 &= (60 - t)(t + 1) + t + (t - 1) + \cdots + 2 + 1 \\
234 &= (60 - t)(t + 1) + \tfrac{1}{2}t(t + 1) \\
468 &= 2(60 - t)(t + 1) + t(t + 1) \\
468 &= (120 - 2t)(t + 1) + t(t + 1) \\
468 &= (120 - t)(t + 1)
\end{aligned}$$

When $t = 3$, the right side is equal to $117 \times 4$ which equals 468.

Expanding, we obtain $468 = -t^2 + 119t + 120$ or $t^2 - 119t + 348 = 0$.

Factoring, we obtain $(t - 3)(t - 116) = 0$.

Since $t$ is between 0 and 15, inclusive, then $t = 3$.

Since $t$ is 3, the greatest integer less than or equal to $\dfrac{120}{x} - 8$ is 3.

This means that $\dfrac{120}{x} - 8 \geq 3$ and $\dfrac{120}{x} - 8 < 4$.

Thus, $\frac{120}{x} \geq 11$ and $\frac{120}{x} < 12$, which tells us that $10 < x \leq \frac{120}{11}$, which is the range of possible values for $x$.