The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Hypatia Contest

April 2021
(in North America and South America)

April 2021
(outside of North America and South America)

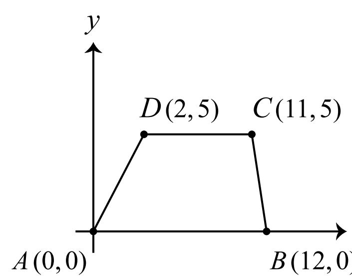
Solutions

1. (a) The total cost to rent a car is \$180.00.
If 4 people rent a car, the cost per person is $\frac{\$180.00}{4} = \$45.00$.

(b) Since the members of the group equally share the total cost to rent the vehicle, the smaller the group, the greater the cost per person.
To rent an SUV, the smallest group size required is 5 passengers and the total cost is \$200.00.
Thus, the maximum possible cost per person to rent an SUV is $\frac{\$200.00}{5} = \$40.00$.

(c) Let the total cost to rent a van be v.
When renting a van, the maximum possible cost per person occurs when the number of passengers is 9 (the fewest number possible), and so this maximum cost is $\frac{v}{9}$.
The minimum possible cost per person occurs when the number of passengers is 12 (the greatest number possible), and so this minimum cost is $\frac{v}{12}$.
Then, $\frac{v}{9} - \frac{v}{12} = \$6.00$ or $\frac{4v - 3v}{36} = \$6.00$, and so $v = \$6.00 \times 36$.
Thus, the total cost to rent a van is \$216.00.

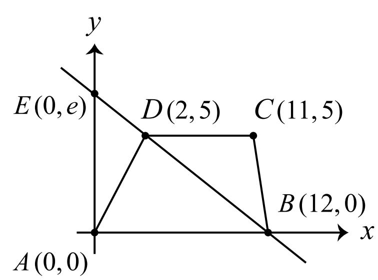
2. (a) Trapezoid $ABCD$ is drawn, as shown.
The slope of line segments AB and CD are each zero and thus they are parallel.
The length of AB is the difference between the x-coordinates of A and B, or 12.
The length of CD is the difference between the x-coordinates of C and D, or $11 - 2 = 9$.



The height of the trapezoid is equal to the vertical distance between AB and CD, which is 5.
The area of trapezoid $ABCD$ is $\frac{5}{2}(AB + CD)$ or $\frac{5}{2}(21) = \frac{105}{2}$.

(b) The line passing through B and D intersects the y-axis at E. Let the coordinates of E be $(0, e)$, as shown.
The slope of the line through B and D is $\frac{5-0}{2-12} = -\frac{1}{2}$.

Solution 1
Since E, D and B lie on the same line, then the slope of ED is equal to the slope of BD.



Equating slopes, we get $\frac{e-5}{0-2} = -\frac{1}{2}$ or $\frac{e-5}{2} = \frac{1}{2}$, and so $e - 5 = 1$ or $e = 6$.
Thus, point E has coordinates $(0, 6)$.

Solution 2
The line passing through $B(12, 0)$ and $D(2, 5)$ has slope $-\frac{1}{2}$, and thus has equation $y - 5 = -\frac{1}{2}(x - 2)$.
Rearranging, we get $y - 5 = -\frac{1}{2}x + 1$ or $y = -\frac{1}{2}x + 6$.
Since this line has y-intercept 6, then point E has coordinates $(0, 6)$.

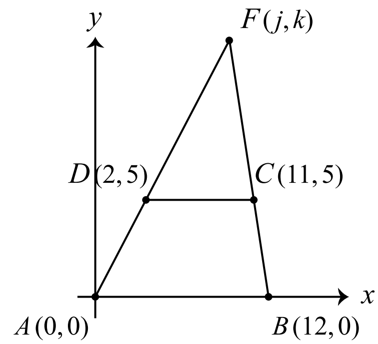
Solution 3

The line passing through $B(12,0)$ and $D(2,5)$ has slope $-\frac{1}{2}$, and thus has equation $y-5=-\frac{1}{2}(x-2)$.

This line passes through $E(0,e)$, and so $e-5=-\frac{1}{2}(0-2)$ or $e=1+5=6$.

Thus, point E has coordinates $(0,6)$.

(c) Sides AD and BC are extended to intersect at F, as shown.



Solution 1

Let the coordinates of F be (j,k).
Since A, D and F lie on the same line, then the slope of AD is equal to the slope of AF.
Equating slopes, we get $\frac{5}{2}=\frac{k}{j}$ or $k=\frac{5}{2}j$.
Since B, C and F lie on the same line, then the slope of BC is equal to the slope of BF.

Equating slopes, we get $\frac{5-0}{11-12}=\frac{k-0}{j-12}$ or $-5=\frac{k}{j-12}$, and so $k=-5(j-12)$.

Substituting $k=\frac{5}{2}j$, we get $\frac{5}{2}j=-5(j-12)$ or $j=-2(j-12)$, and so $3j=24$ or $j=8$.

When $j=8$, $k=\frac{5}{2}(8)=20$, and so F has coordinates $(8,20)$.

Solution 2

The line passing through $A(0,0)$ and $D(2,5)$ has slope $\frac{5}{2}$ and y-intercept 0, and thus has equation $y=\frac{5}{2}x$.
The line passing through $B(12,0)$ and $C(11,5)$ has slope -5 and thus has equation $y=-5(x-12)$.
These two lines intersect at F, and so the coordinates of F can be determined by solving the equation $\frac{5}{2}x=-5(x-12)$. Solving, we get $x=-2(x-12)$ or $3x=24$, and so $x=8$.

When $x=8$, $y=\frac{5}{2}(8)=20$, and so F has coordinates $(8,20)$.

(d) Let P have coordinates (r,s).
Assume AB is the base of $\triangle PAB$.

In this case, if the height of $\triangle PAB$ is h, then the area of $\triangle PAB$ is $\frac{1}{2}(AB)h=6h$.

The area of $\triangle PAB$ is 42, and so $6h=42$ or $h=7$.
That is, $P(r,s)$ is located a vertical distance of 7 units from the line through A and B, or 7 units from the x-axis.
There are two possibilities: $P(r,s)$ is located 7 units above the x-axis, and thus lies on the horizontal line $y=7$, or $P(r,s)$ is located 7 units below the x-axis, and thus lies on the horizontal line $y=-7$.
In the first case, P has coordinates $(r,7)$ and in the second case, P has coordinates $(r,-7)$.
Recall that P lies on the line passing through B and D.

The line passing through $B(12,0)$ and $D(2,5)$ has slope $-\frac{1}{2}$, and thus has equation $y - 5 = -\frac{1}{2}(x - 2)$.

If $P(r, 7)$ lies on this line, then $7 - 5 = -\frac{1}{2}(r - 2)$ or $-4 = r - 2$, and so $r = -2$.

Similarly, if $P(r, -7)$ lies on this line, then $-7 - 5 = -\frac{1}{2}(r - 2)$ or $24 = r - 2$, and so in this case, $r = 26$.

The points P that lie on the line passing through B and D, so that the area of $\triangle PAB$ is 42, are $(-2, 7)$ and $(26, -7)$.

3. (a) Since $a_n = 2^n$ for $n \geq 1$, then $a_5 = 2^5 = 32$.
Since $b_2 = 1, b_3 = 3$, and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 3$, then $b_4 = b_3 + 2b_2 = 3 + 2(1) = 5$ and $b_5 = b_4 + 2b_3 = 5 + 2(3) = 11$.
Therefore, $a_5 = 32$ and $b_5 = 11$.

(b) Since $b_1 = p \cdot (a_1) + q \cdot (-1)^1$ and $a_1 = 2$, then $b_1 = 2p - q$.
From the definition of sequence B, we know $b_1 = 1$, and so $2p - q = 1$.
Since $b_2 = p \cdot (a_2) + q \cdot (-1)^2$ and $a_2 = 2^2 = 4$, then $b_2 = 4p + q$.
From the definition of sequence B, we know $b_2 = 1$, and so $4p + q = 1$.
This gives two equations in two unknowns, p and q.
Adding these two equations, we get $6p = 2$, and so $p = \frac{1}{3}$.
Substituting, we get $2\left(\frac{1}{3}\right) - q = 1$ or $q = \frac{2}{3} - 1 = -\frac{1}{3}$.
Thus, the real numbers p and q for which $b_n = p \cdot (a_n) + q \cdot (-1)^n$ for all $n \geq 1$ are $p = \frac{1}{3}$ and $q = -\frac{1}{3}$.

(c) Using algebraic manipulation, and the definitions $a_n = 2^n$ and $b_n = \frac{1}{3}(a_n) - \frac{1}{3}(-1)^n$, each for $n \geq 1$, we obtain the following equivalent equations,

$$\begin{aligned} S_n &= b_1 + b_2 + b_3 + \cdots + b_n \\ &= \left(\tfrac{1}{3}(a_1) - \tfrac{1}{3}(-1)\right) + \left(\tfrac{1}{3}(a_2) - \tfrac{1}{3}(-1)^2\right) + \left(\tfrac{1}{3}(a_3) - \tfrac{1}{3}(-1)^3\right) + \cdots + \left(\tfrac{1}{3}(a_n) - \tfrac{1}{3}(-1)^n\right) \\ &= \tfrac{1}{3}(a_1 + a_2 + a_3 + \cdots + a_n) - \tfrac{1}{3}((-1) + (-1)^2 + (-1)^3 + \cdots + (-1)^n) \\ &= \tfrac{1}{3}(2 + 2^2 + 2^3 + \cdots + 2^n) - \tfrac{1}{3}(-1 + 1 - 1 + \cdots + (-1)^n) \end{aligned}$$

Next, we consider each of the two expressions within parentheses, separately.
The expression $2 + 2^2 + 2^3 + \cdots + 2^n$ is the sum of n terms of a geometric sequence with first term $a = 2$ and common ratio $r = 2$.
Thus, $2 + 2^2 + 2^3 + \cdots + 2^n = 2\left(\frac{1 - 2^n}{1 - 2}\right) = -2(1 - 2^n)$.
The expression $-1 + 1 - 1 + \cdots + (-1)^n$ is an alternating sum of the terms -1 and 1.
This simplifies to 0 if there are an even number of terms, that is, if n is even, and simplifies to -1 if n is odd.
Summarizing, we have

$$S_n = \begin{cases} \frac{1}{3}(-2(1 - 2^n)) \text{ , if } n \text{ is even} \\ \frac{1}{3}(-2(1 - 2^n)) + \frac{1}{3} \text{ , if } n \text{ is odd} \end{cases}$$

and simplifying, we get

$$S_n = \begin{cases} \frac{2}{3}(2^n - 1) \text{ , if } n \text{ is even} \\ \frac{2}{3}(2^n - 1) + \frac{1}{3} \text{ , if } n \text{ is odd} \end{cases}$$

We want the smallest positive integer n that satisfies $S_n \geq 16^{2021}$ and note that the value of S_n increases as n increases.
Since $16 = 2^4$, then $16^{2021} = (2^4)^{2021} = 2^{8084}$ and so we want the smallest positive integer n that satisfies $S_n \geq 2^{8084}$.

When n is even, we get

$$\begin{aligned} \tfrac{2}{3}(2^n - 1) &\geq 2^{8084} \\ 2^n - 1 &\geq 3 \cdot 2^{8083} \\ 2^n &\geq 3 \cdot 2^{8083} + 1 \end{aligned}$$

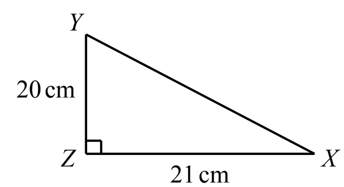
When n is odd, we get

$$\begin{aligned} \tfrac{2}{3}(2^n - 1) + \tfrac{1}{3} &\geq 2^{8084} \\ \tfrac{2}{3}(2^n - 1) &\geq 2^{8084} - \tfrac{1}{3} \\ 2^n - 1 &\geq 3 \cdot 2^{8083} - \tfrac{1}{2} \\ 2^n &\geq 3 \cdot 2^{8083} + \tfrac{1}{2} \end{aligned}$$

Since 2^n is an even integer, $3 \cdot 2^{8083} + 1$ is an odd integer and $3 \cdot 2^{8083} + \frac{1}{2}$ is between an even integer and an odd integer, and thus the inequalities $2^n \geq 3 \cdot 2^{8083} + 1$ and $2^n \geq 3 \cdot 2^{8083} + \frac{1}{2}$ are both equivalent to saying $2^n > 3 \cdot 2^{8083}$.

Since $3 \cdot 2^{8083} > 2 \cdot 2^{8083}$, then simplifying, we get $3 \cdot 2^{8083} > 2^{8084}$.
Thus, we want the smallest positive integer n that satisfies $2^n > 3 \cdot 2^{8083} > 2^{8084}$.
When $n \leq 8084$, this inequality is not true.
When $n = 8085$, we get $2^{8085} = 2^2 \cdot 2^{8083} = 4 \cdot 2^{8083}$ which is greater than $3 \cdot 2^{8083}$, as required.
Thus, the smallest positive integer n that satisfies $S_n \geq 16^{2021}$ is $n = 8085$.

4. (a) In $\triangle XYZ$, $x = 20$, $y = 21$, and $\angle XZY = 90°$, as shown. Using the Pythagorean Theorem, we get $z = \sqrt{20^2 + 21^2} = 29$ (since $z > 0$).



The value of A is

$$A = \frac{1}{2}(y)(x) = \frac{1}{2}(21)(20) = 210$$

The value of P is

$$P = z + x + y = 29 + 20 + 21 = 70$$

(b) When $A = 336$, we get $\frac{1}{2}xy = 336$, and so $xy = 672$.
By the Pythagorean Theorem, $x^2 + y^2 = 50^2$, which when manipulated algebraically gives the following equivalent equations

$$\begin{aligned} x^2 + y^2 &= 2500 \\ (x + y)^2 - 2xy &= 2500 \\ (x + y)^2 &= 2500 + 2xy \\ (x + y)^2 &= 2500 + 2(672) \\ (x + y)^2 &= 3844 \end{aligned}$$

Since $x + y > 0$, then $x + y = \sqrt{3844} = 62$.
Thus, we get $P = x + y + z = 62 + 50 = 112$.
(The triangle satisfying these conditions has side lengths 14 cm, 48 cm and 50 cm.)

(c) Since $A = 3P$, we get $\frac{1}{2}xy = 3(x + y + z)$, and so $xy = 6(x + y + z)$.
When $xy = 6(x + y + z)$ is manipulated algebraically, we get the following equivalent equations

$$\begin{aligned}
xy &= 6(x+y+z) \\
xy - 6x - 6y &= 6z \\
(xy - 6x - 6y)^2 &= (6z)^2 \\
(xy)^2 - 12x(xy) - 12y(xy) + 72xy + 36x^2 + 36y^2 &= 36z^2 \\
(xy)^2 - 12x(xy) - 12y(xy) + 72xy + 36x^2 + 36y^2 &= 36(x^2+y^2) \quad (\text{since } x^2+y^2=z^2) \\
xy(xy - 12x - 12y + 72) &= 0 \\
xy - 12x - 12y + 72 &= 0 \quad (\text{since } xy \neq 0) \\
x(y-12) - 12y &= -72 \\
x(y-12) - 12y + 144 &= -72 + 144 \\
x(y-12) - 12(y-12) &= 72 \\
(x-12)(y-12) &= 72
\end{aligned}$$

Since x and y are positive integers, then $x - 12$ and $y - 12$ are a factor pair of 72.
The product $(x-12)(y-12)$ is positive (since $72 > 0$), and thus $x-12 < 0$ and $y-12 < 0$ or $x - 12 > 0$ and $y - 12 > 0$.
If $x - 12 < 0$ and $y - 12 < 0$, then $x < 12$ and $y < 12$.
There are exactly two Pythagorean triples (x, y, z) in which $x < 12$ and $y < 12$.
In the first case, $(x, y, z) = (3, 4, 5)$, which gives $A = \frac{1}{2}(3)(4) = 6$, $P = 3+4+5 = 12$, and so $A \neq 3P$.
In the second case, $(x, y, z) = (6, 8, 10)$, which gives $A = \frac{1}{2}(6)(8) = 24$, $P = 6+8+10 = 24$, and so $A \neq 3P$.
Therefore, $x - 12 > 0$ and $y - 12 > 0$, and so x and y are each greater than 12.
In the table below, we use the positive factor pairs of 72 to determine all possible integer values of x, y and z.
Further, we initially make the assumption that $x \leq y$, recognizing that by the symmetry of the equation, the values of x and y may be interchanged with one another and doing so gives the same value for z and the same triangle.

Factor pair	$x-12$	$y-12$	x	y	$z=\sqrt{x^2+y^2}$	(x,y,z)
1 and 72	1	72	13	84	85	$(13, 84, 85)$ or $(84, 13, 85)$
2 and 36	2	36	14	48	50	$(14, 48, 50)$ or $(48, 14, 50)$
3 and 24	3	24	15	36	39	$(15, 36, 39)$ or $(36, 15, 39)$
4 and 18	4	18	16	30	34	$(16, 30, 34)$ or $(30, 16, 34)$
6 and 12	6	12	18	24	30	$(18, 24, 30)$ or $(24, 18, 30)$
8 and 9	8	9	20	21	29	$(20, 21, 29)$ or $(21, 20, 29)$

It is worth noting that instead of factoring the equation $xy - 12x - 12y + 72 = 0$ as we did, we could have chosen to rewrite it as

$$\begin{aligned} x(y-12) &= 12y-72 \\ x &= \frac{12y-72}{y-12} \\ x &= \frac{12(y-12)+144-72}{y-12} \\ x &= 12+\frac{72}{y-12} \end{aligned}$$

and considered that since x is a positive integer, then $y-12$ is a divisor of 72.

(d) Since $A = kP$, we get $\frac{1}{2}xy = k(x+y+z)$, and so $xy = 2k(x+y+z)$.
When $xy = 2k(x+y+z)$ is manipulated algebraically, we get the following equivalent equations

$$\begin{aligned} xy &= 2k(x+y+z) \\ xy-2kx-2ky &= 2kz \\ (xy-2kx-2ky)^2 &= (2kz)^2 \\ (xy)^2-4kx(xy)-4ky(xy)+8k^2xy+4k^2x^2+4k^2y^2 &= 4k^2z^2 \\ (xy)^2-4kx(xy)-4ky(xy)+8k^2xy+4k^2x^2+4k^2y^2 &= 4k^2(x^2+y^2) \quad (\because x^2+y^2=z^2) \\ xy(xy-4kx-4ky+8k^2) &= 0 \\ xy-4kx-4ky+8k^2 &= 0 \quad (\text{since } xy \neq 0) \\ x(y-4k)-4ky &= -8k^2 \\ x(y-4k)-4ky+16k^2 &= -8k^2+16k^2 \\ x(y-4k)-4k(y-4k) &= 8k^2 \\ (x-4k)(y-4k) &= 8k^2 \end{aligned}$$

Since x, y and k are positive integers, then $x-4k$ and $y-4k$ are a factor pair of $8k^2$.
We begin by assuming that $k = 2$.
Substituting, we get $(x-8)(y-8) = 32$.
The product $(x-8)(y-8)$ is positive (since $32 > 0$), and thus $x-8 < 0$ and $y-8 < 0$ or $x-8 > 0$ and $y-8 > 0$.
If $x-8 < 0$ and $y-8 < 0$, then $x < 8$ and $y < 8$ which is not possible since $P = 510$, and so $x-8 > 0$ and $y-8 > 0$.
If $(x-8)(y-8) = 32$ and $x \leq y$, then $x-8$ is equal to $1, 2$ or 4, which gives $x = 9, 10, 12$ and $y-8$ is equal to $32, 16, 8$, and so $y = 40, 24, 16$, respectively.
Using the Pythagorean Theorem, we get $z = 41, 26, 20$, respectively.
For each of the three possibilities, $P = x+y+z \neq 510$ and so we conclude $k \neq 2$.
It can similarly be shown that k cannot equal 3, and thus $k \geq 5$ (since k is a prime number).
Since k is a prime number and $k \geq 5$, the positive factor pairs of $8k^2$ are

$$(1, 8k^2), (2, 4k^2), (4, 2k^2), (8, k^2), (k, 8k), (2k, 4k)$$

and the negative factor pairs of $8k^2$ are

$$(-1, -8k^2), (-2, -4k^2), (-4, -2k^2), (-8, -k^2), (-k, -8k), (-2k, -4k)$$

If for example $x - 4k = -1$ and $y - 4k = -8k^2$, then $y = 4k - 8k^2$ which is less than zero for all $k \geq 5$.
This is not possible since $y > 0$.
Assuming $x \leq y$, it can similarly be shown that when $x - 4k$ and $y - 4k$ are equal to a negative factor pair of $8k^2$, then $y \leq 0$ for all values of $k \geq 5$.
Thus, $x - 4k$ and $y - 4k$ must equal a positive factor pair of $8k^2$.

Beginning with the fact that the perimeter of the triangle is 510 cm, we get the following equivalent equations

$$\begin{aligned}
x + y + z &= 510 \\
x + y &= 510 - z \\
x^2 + 2xy + y^2 &= 510^2 - 1020z + z^2 \quad \text{(squaring both sides)} \\
2xy &= 510^2 - 1020z \quad (\text{since } x^2 + y^2 = z^2) \\
4A &= 510^2 - 1020z \quad (A = \tfrac{1}{2}xy \text{ and so } 4A = 2xy) \\
4(510k) &= 510^2 - 1020z \quad (\text{since } A = kP \text{ and } P = 510) \\
2k &= 255 - z \\
2k &= 255 - (510 - x - y) \\
x + y - 2k &= 255
\end{aligned}$$

From the first factor pair, we get $x - 4k = 1$ and $y - 4k = 8k^2$, and so $(x, y) = (1 + 4k, 8k^2 + 4k)$ (assuming $x \leq y$).
Substituting $x = 1 + 4k$ and $y = 8k^2 + 4k$ into $x + y - 2k = 255$ and simplifying, we get $8k^2 + 6k = 254$ or $k(4k + 3) = 127$, which has no solutions since 127 is a prime number.

We continue our analysis of the remaining 5 factor pairs in the table below.
As before, we make the assumption that $x \leq y$, recognizing that the values of x and y may be interchanged with one another and doing so gives the same value(s) for k.

Factor pair	x	y	$x + y - 2k = 255$ simplified	Value(s) of k
$2, 4k^2$	$2 + 4k$	$4k^2 + 4k$	$4k^2 + 6k = 253$	No k (LS is even and the RS is odd)
$4, 2k^2$	$4 + 4k$	$2k^2 + 4k$	$2k^2 + 6k = 251$	No k (LS is even and the RS is odd)
$8, k^2$	$8 + 4k$	$k^2 + 4k$	$k(k + 6) = 247$	$k = 13$
$k, 8k$	$5k$	$12k$	$15k = 255$	$k = 17$
$2k, 4k$	$6k$	$8k$	$12k = 255$	No k (LS is even and the RS is odd)

Therefore, the values of k which satisfy the given conditions are $k = 13$ and $k = 17$.