Hypatia Contest

(Grade 11)

April 2021
(in North America and South America)

April 2021
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes $\qquad$ ©2021 University of Waterloo

Do not open this booklet until instructed to do so.

Number of questions: 4 $\qquad$ **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 💡

   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

2. **FULL SOLUTION** parts indicated by ✎

   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

   - Extra paper for your finished solutions must be supplied by your supervising teacher and inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
   - Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

1. A company rents out various sized passenger vehicles according to the following table. For example, a group of $5, 6, 7$, or 8 people would need to rent a sports utility vehicle (SUV), which has a total cost of $200.00. Unfortunately, the total cost to rent a van is missing from the table. In each case, the members of the group equally share the total cost to rent the vehicle.

Vehicle	Number of Passengers Required	Total Cost
Car	1 to 4	$180.00
SUV	5 to 8	$200.00
Van	9 to 12	

   💡 (a) If 4 people rent a car, what is the cost per person?

   💡 (b) If a group rents an SUV, what is the maximum possible cost per person?

   ✒ (c) When a van is rented, the difference between the maximum cost per person and the minimum cost per person is $6.00. Determine the total cost to rent a van.

2. Trapezoid $ABCD$ has vertices $A(0,0)$, $B(12,0)$, $C(11,5)$, $D(2,5)$.

   💡 (a) What is the area of trapezoid $ABCD$?

   💡 (b) The line passing through B and D intersects the y-axis at the point E. What are the coordinates of E?

   ✒ (c) The sides AD and BC are extended to intersect at the point F. Determine the coordinates of F.

   ✒ (d) Determine all possible points P that lie on the line passing through B and D, so that the area of $\triangle PAB$ is 42.

3. The sequence A, with terms $a_1, a_2, a_3, \ldots$, is defined by
$$a_n = 2^n, \text{ for } n \geq 1.$$
The sequence B, with terms $b_1, b_2, b_3, \ldots$, is defined by
$$b_1 = 1, \ b_2 = 1, \text{ and } b_n = b_{n-1} + 2b_{n-2}, \text{ for } n \geq 3.$$
For example, $b_3 = b_2 + 2b_1 = 1 + 2(1) = 3$.

In this question, the following facts may be helpful:

- A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant called the common ratio. For example, $3, 6, 12$ is a geometric sequence with three terms and common ratio 2.

- The sum of the first n terms of a geometric sequence with first term a, and common ratio $r \neq 1$, equals $a \left(\dfrac{1 - r^n}{1 - r} \right)$.

(a) What are the 5th terms for each sequence? That is, what are the values of a_5 and b_5?

(b) For some real numbers p and q, $b_n = p \cdot (a_n) + q \cdot (-1)^n$ for all $n \geq 1$. (You do not need to show this.) What are the values of p and q?

(c) Let S_n be the sum of the first n terms in sequence B. That is, $S_n = b_1 + b_2 + b_3 + \cdots + b_n$. Determine the smallest positive integer n that satisfies $S_n \geq 16^{2021}$.

4. In $\triangle XYZ$, the measure of $\angle XZY$ is $90°$. Also, $YZ = x$ cm, $XZ = y$ cm, and hypotenuse XY has length z cm. Further, the perimeter of $\triangle XYZ$ is P cm and the area of $\triangle XYZ$ is A cm^2.

(a) If $x = 20$ and $y = 21$, what are the values of A and P?

(b) If $z = 50$ and $A = 336$, what is the value of P?

(c) Determine all possible integer values of x, y and z for which $A = 3P$.

(d) Suppose that x, y and z are integers, that $P = 510$, and that $A = kP$ for some prime number k. Determine all possible values of k.

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Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2021.

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