The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2021 Gauss Contests

(Grades 7 and 8)

Wednesday, May 12, 2021
(in North America and South America)

Thursday, May 13, 2021
(outside of North America and South America)

Solutions

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Grade 7

1. Arranging the five numbers from largest to smallest, we get $10\,000, 1000, 100, 10, 1$.
   The middle number is 100.

   Answer: (D)

2. Each side of the square has length 5 cm.
   The perimeter of the square is 4×5 cm $= 20$ cm.

   Answer: (A)

3. The right side of the equation is $10 + 20 = 30$.
   The equation is true when the left side is also equal to 30.
   Since $5 + 25 = 30$, the value that goes in the box to make the equation true is 25.

   Answer: (E)

4. Reading from the graph, Dan spent 6 hours on homework, Joe spent 3 hours, Bob spent 5 hours, Susie spent 4 hours, and Grace spent 1 hour.
   Adding their times together, Bob and Grace spent the same amount of time on homework as Dan.

   Answer: (C)

5. Each of the five fractions is positive and so the smallest of these fractions is the fraction that is closest to 0.
   Since each fraction has a numerator equal to 1, the smallest of these fractions is the one with the largest denominator.
   Of those given, the fraction that is closest to 0 is thus $\frac{1}{9}$.

   Answer: (E)

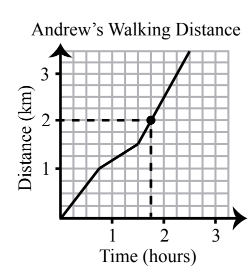
6. If the bag contained a total of 6 candies and exactly 5 of these candies were red, then the probability of Judith choosing a red candy from the bag would be $\frac{5}{6}$.
   Therefore, the total number of candies in the bag could be 6.
   Can you explain why each of the other four answers is not possible?

   Answer: (D)

7. Each point that lies to the right of the y-axis has an x-coordinate that is positive.
   Each point that lies below the x-axis has a y-coordinate that is negative.
   Since $P(x, y)$ lies to the right of the y-axis and below the x-axis, then the value of x is positive and the value of y is negative.

   Answer: (B)

8. Begin by locating 2 km on the vertical (Distance) axis.
   Next, locate the point on the line graph for which Andrew's distance walked is 2 km, as shown.
   The time in hours corresponding to this point is $\frac{3}{4}$ of an hour greater than 1 hour.
   Since $\frac{1}{4}$ of an hour is equal to 15 minutes, then $\frac{3}{4}$ of an hour is 45 minutes, and so it takes Andrew 1 hour, 45 minutes to walk the first 2 km.

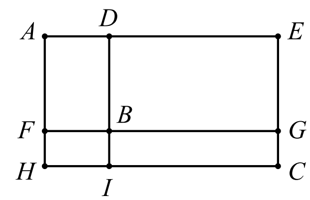


   Answer: (C)

9. The five numbers $5, 6, 7, 8, 9$ repeat to form the pattern shown.
Thus, the 5^{th} number in the pattern is 9, the 10^{th} number in the pattern is 9, the 15^{th} number in the pattern is 9, and so on.
Since 220 is a multiple of 5, the 220^{th} number in the pattern is also a 9 and so the 221^{st} number in the pattern is 5.

Answer: (A)

10. We begin by labelling additional points, as shown.
Beginning at A, the ant may travel right (to D) or down (to F).



Assume the ant begins by travelling right, to D.
From D, if the ant continues travelling right (to E) the path cannot pass through B (since the ant can travel only right or down).
Thus, from D, the ant must travel down to B.

From B, there are two paths that end at C, one travelling right to G and then down to C, and another travelling down to I and then right to C.
Therefore, there are two paths in which the ant begins by travelling right: $A - D - B - G - C$ and $A - D - B - I - C$.

Assume the ant begins by travelling down, to F.
From F, if the ant continues travelling down (to H) the path cannot pass through B (since the ant can travel only right or down). Thus, from F, the ant must travel right to B.
From B, there are two paths that end at C, as discussed above.
Therefore, there are two paths in which the ant begins by travelling down: $A - F - B - G - C$ and $A - F - B - I - C$.

There are 4 different paths from A to C that pass through B.

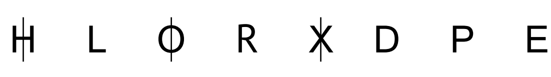
Answer: (C)

11. *Solution 1*
Writing the numbers that appear in the list, we get
$$4, 11, 18, 25, 32, 39, 46, 53, \ldots$$
Of the given answers, the number that appears in Laila's list is 46.

Solution 2
Laila begins her list at 4 and each new number is 7 more than the previous number.
Therefore, each of the numbers in her list will be 4 more than a multiple of 7.
Since 42 is a multiple of 7 ($6 \times 7 = 42$), then $42 + 4 = 46$ will appear in Laila's list.

Answer: (B)

12. If a letter is folded along its vertical line of symmetry, both halves of the letter would match exactly.
Three of the given letters, H, O and X, have a vertical line of symmetry, as shown.



Answer: (C)

13. Since $\triangle BCE$ is equilateral, then each of its three interior angles measures 60°.
Vertically opposite angles are equal in measure, and so $\angle DEA = \angle BEC = 60^\circ$.
In $\triangle ADE$, the sum of the three interior angles is 180°.
Thus, $x^\circ + 90^\circ + 60^\circ = 180^\circ$ or $x + 150 = 180$ and so $x = 30$.

Answer: (E)

14. Given three consecutive integers, the smallest integer is one less than the middle integer and the largest integer is one more than the middle integer.
For example, 10, 11 and 12 are three consecutive integers, and 10 is one less than the middle integer 11, and 12 is one more than 11.
So, the sum of the smallest and largest integers is twice the middle integer.
(In the example, $10 + 12 = 2 \times 11$.)
Then, the sum of three consecutive integers is equal to 3 times the middle integer and so the sum of three consecutive integers is a multiple of 3.
Of the answers given, the only number that is a multiple of 3 is 21.
Alternately, we could use trial and error to solve this problem to find that $6 + 7 + 8 = 21$.

ANSWER: (D)

15. There is no integer greater than 13931 and less than 14000 that is a palindrome. (You should consider why this is true before reading on.)
Let the next palindrome greater than 13931 be N.
We proceed under the assumption that N is between 14000 and 15000 and will show that this assumption is correct.
A 5-digit palindrome is a number of the form $abcba$. That is, the ten thousands digit, a, must equal the ones digit and the thousands digit, b, must equal the tens digit.
Since N is at least 14000, the smallest possible value of a (the ten thousands digit) is 1.
Since the smallest possible value of a is 1 and N is at least 14000, the smallest possible value of b (the thousands digit) is 4.
Thus N is a number of the form $14c41$.
Letting the hundreds digit, c, be as small as possible we get that N is 14041 and has digit sum $1 + 4 + 0 + 4 + 1 = 10$.

ANSWER: (D)

16. The positive factors of 14 are $1, 2, 7$, and 14.
The positive factors of 21 are $1, 3, 7$, and 21.
The positive factors of 28 are $1, 2, 4, 7, 14$, and 28.
The positive factors of 35 are $1, 5, 7$, and 35.
The positive factors of 42 are $1, 2, 3, 6, 7, 14, 21$, and 42.
There are 3 numbers in the list (14, 21 and 35) that have exactly 4 positive factors.

ANSWER: (C)

17. The percentage discount of the third price off the original price does not depend on the original price of the shirt.
That is, we may choose an original price for the shirt and calculate the combined percentage discount.
Since the discounts are given as percentages, letting the original price of the shirt be \$100 might make the calculations simpler.
If the original price of the shirt is \$100 and this price is reduced by 50%, the discounted price is half of \$100 or \$50.
A further 40% reduction on \$50 is equal to a $\frac{40}{100} \times \$50 = 0.40 \times \$50 = \$20$ discount.
After both price reductions, the \$100 shirt is priced at $\$50 - \$20 = \$30$ and thus the total discount is $\$100 - \$30 = \$70$.
The original price of the shirt was \$100, the final discounted price is \$70 less, and so the discount of the third price off the original price is $\frac{\$70}{\$100} \times 100\% = 70\%$.

ANSWER: (C)

18. The perimeter of $\triangle ABC$ is equal to $AB + BC + CA$ or $AB + BM + MC + CA$.
Since $AB = CA$ and $BM = MC$, then half of the perimeter of $\triangle ABC$ is equal to $AB + BM$.
The perimeter of $\triangle ABC$ is 64 and so $AB + BM = \frac{1}{2} \times 64 = 32$.
The perimeter of $\triangle ABM$ is 40, and so $AM + AB + BM = 40$.
Since $AB + BM = 32$, then $AM = 40 - 32 = 8$.

Answer: (B)

19. We begin by calling the missing digits A and B, as shown. Each of the digits A and B is chosen from the digits 1 to 9 and $A \neq B$.

$$\begin{array}{r} 5\,\boxed{A} \\ -\ \boxed{B}\,5 \\ \hline \end{array}$$

The top 2-digit number $5A$ is at most 59.
Thus, B cannot equal 6, 7, 8, or 9 since the result of subtracting the bottom number from the top would be negative.
If $B = 5$, then the bottom number is 55 and for the result to be positive, A could equal 6, 7, 8, or 9 (with the results being 1, 2, 3, and 4 respectively).
Thus, there are 4 possible positive results in this case.
If $B = 4$, then the bottom number is 45 and for the result to be positive, A could equal each of the digits from 1 to 9 with the exception of 4 (since $A \neq B$).
In this case, the results of subtracting the bottom number from the top are 6, 7, 8, 10, 11, 12, 13, and 14 respectively. Thus, there are 8 possible positive results when $B = 4$.
If $B = 3$, then the bottom number is 35 and for the result to be positive, A could equal 1, 2, 4, 5, 6, 7, 8, or 9 (with the results being 16, 17, 19, 20, 21, 22, 23, and 24 respectively).
Thus, there are 8 possible positive results when $B = 3$.
Similarly, when $B = 2$ there are 8 possible positive results and when $B = 1$ there are 8 possible positive results (and each of these results is different from any of the other results).
In total, the number of possible results that are positive is $4 + (8 \times 4) = 36$.

Answer: (A)

20. The table below shows the possible sums when two standard dice are rolled.
Each sum in bold is equal to a prime number.

Number on the First Die

Number on the Second Die

	1	2	3	4	5	6
1	**2**	**3**	4	**5**	6	**7**
2	**3**	4	**5**	6	**7**	8
3	4	**5**	6	**7**	8	9
4	**5**	6	**7**	8	9	10
5	6	**7**	8	9	10	**11**
6	**7**	8	9	10	**11**	12

Looking at the table above, the total number of possible outcomes is $6 \times 6 = 36$.
The total number of outcomes for which the sum is a prime number is 15.
The probability that the sum of the numbers on the top faces is a prime number is $\frac{15}{36} = \frac{5}{12}$.

Answer: (A)

21. We begin by considering cases in which 1 is subtracted from numbers that start with one and are followed by smaller numbers of zeros.

$$\begin{array}{r} 10 \\ -1 \\ \hline 9 \end{array} \qquad \begin{array}{r} 100 \\ -1 \\ \hline 99 \end{array} \qquad \begin{array}{r} 1000 \\ -1 \\ \hline 999 \end{array} \qquad \begin{array}{r} 10\,000 \\ -1 \\ \hline 9999 \end{array} \qquad \begin{array}{r} 100\,000 \\ -1 \\ \hline 99\,999 \end{array}$$

In the examples above, each result consists of 9s only and the number of 9s is equal to the number of zeros in the original number. Can you explain why this pattern continues as we increase the number of zeros?
Since each of the digits in the result is a 9 and the sum of these digits is 252, then the number of 9s in the result is equal to $\frac{252}{9} = 28$.
The number of zeros in the original number equals the number of 9s in the result, which is 28.

Answer: (B)

22. The perimeter of Figure 1 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 4 rectangle side lengths of 5 cm (each of which is vertical).
Thus, the perimeter of Figure 1 is $(4 \times 10 \text{ cm}) + (4 \times 5 \text{ cm}) = 40 \text{ cm} + 20 \text{ cm} = 60 \text{ cm}$.
The perimeter of Figure 2 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 6 rectangle side lengths of 5 cm (each of which is vertical).
Thus, the perimeter of Figure 2 is $(4 \times 10 \text{ cm}) + (6 \times 5 \text{ cm}) = 40 \text{ cm} + 30 \text{ cm} = 70 \text{ cm}$.
The perimeter of Figure 3 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 8 rectangle side lengths of 5 cm (each of which is vertical).
Thus, the perimeter of Figure 3 is $(4 \times 10 \text{ cm}) + (8 \times 5 \text{ cm}) = 40 \text{ cm} + 40 \text{ cm} = 80 \text{ cm}$.
Each figure after Figure 1 is formed by joining two rectangles to the bottom of the previous figure.
The bottom edge of a figure (consisting of two 10 cm side lengths) is replaced by two 10 cm lengths when the two new rectangles are adjoined.
That is, the addition of two rectangles does not change the number of 10 cm side lengths contributing to the perimeter of the new figure and so the number of 10 cm lengths remains constant at 4 for each figure.
The addition of two new rectangles does not replace any of the previous 5 cm (vertical) side lengths.
Thus, the addition of the two rectangles does add two 5 cm vertical segments to the previous perimeter, increasing the perimeter of the previous figure by $2 \times 5 \text{ cm} = 10 \text{ cm}$.
That is, the perimeter of Figure 1 is 60 cm, and the perimeter of each new figure is 10 cm greater than the previous figure.
We need to add 10 cm 65 times to get a total of 710 cm (that is, $60 \text{ cm} + 10 \text{ cm} \times 65 = 710 \text{ cm}$).
Thus, Figure 66 has a perimeter of 710 cm, and so $n = 66$.

Answer: (C)

23. To encode a letter, James multiplies its corresponding number by 3 and then subtracts 5, continuing this process a total of n times.
To decode a number, the inverse operations must be performed in the opposite order.
The inverse operation of multiplication is division. The inverse operation of subtraction is addition.
Thus to decode a number, add 5 and then divide by 3, and continue this process a total of n times.

For example, when $n = 1$ the number 4 (corresponding to the letter D) is encoded to $4 \times 3 - 5 = 7$. The number 7 is decoded by adding 5 and then dividing by 3. We may check that this works by noting that $(7 + 5) \div 3 = 4$ (which corresponds to the letter D), as required.
Each letter of James' original message corresponds to a number from 1 to 26, inclusive.
To determine the value of n, we may begin with the four given encoded numbers $(367, 205, 853, 1339)$ and continue to apply the decoding process until the resulting numbers are each equal to a number from 1 to 26, inclusive (since each letter of the original message corresponds to a number from 1 to 26, inclusive).
We show this work in the table below.

	367	205	853	1339
$n = 1$	$(367 + 5) \div 3 = 124$	$(205 + 5) \div 3 = 70$	$(853 + 5) \div 3 = 286$	$(1339 + 5) \div 3 = 448$
$n = 2$	$(124 + 5) \div 3 = 43$	$(70 + 5) \div 3 = 25$	$(286 + 5) \div 3 = 97$	$(448 + 5) \div 3 = 151$
$n = 3$	16	10	34	52
$n = 4$	7	5	13	19
$n = 5$	4	$\frac{10}{3}$	6	8

From the table, the first value of n for which each of the four numbers is from 1 to 26 inclusive, is $n = 4$.
Further, we note that if $n = 5$ the original number corresponding to 205 is $\frac{10}{3}$, which is not possible.
Therefore, the value of n used by James was 4.
(Although the question did not ask for the original message, the letters corresponding to $7, 5, 13, 19$ are G, E, M, S.)

Answer: (C)

24. We begin by considering the prime factors (called the *prime factorization*) of each of the two numbers, 4 and 4620.

$$4 = 2 \times 2$$

$$4620 = 2 \times 2 \times 3 \times 5 \times 7 \times 11$$

Let a pair of positive whole numbers whose greatest common factor is 4 and whose lowest common multiple is 4620 be (a, b).
Since the pair (a, b) has a greatest common factor of 4, then each of a and b is a multiple of 4 and thus 2×2 is included in the prime factorization of each of a and b.
Further, each of a and b cannot have any other prime factors in common otherwise their greatest common factor would be greater than 4.
The lowest common multiple of a and b is 4620 and so each of a and b is less than or equal to 4620.
Further, if for example a has prime factors which are not prime factors of 4620 then 4620 is not a multiple of a.
That is, each of a and b must have prime factors which are chosen from $2, 2, 3, 5, 7$, and 11 only.
Summarizing, a is a positive whole number of the form $2 \times 2 \times m$ and b is a positive whole number of the form $2 \times 2 \times n$, where:

- m and n are positive whole numbers
- the prime factors of m are chosen from $3, 5, 7, 11$ only
- the prime factors of n are chosen from $3, 5, 7, 11$ only
- m and n have no prime factors in common

In the table below, we list the possible values for m and n which give the possible pairs a and b. To ensure that we don't double count the pairs (a, b), we assume that $a \leq b$ and thus $m \leq n$.

m	n	a	b	(a, b)
1	$3 \times 5 \times 7 \times 11$	$2 \times 2 \times 1$	$2 \times 2 \times 3 \times 5 \times 7 \times 11$	$(4, 4620)$
3	$5 \times 7 \times 11$	$2 \times 2 \times 3$	$2 \times 2 \times 5 \times 7 \times 11$	$(12, 1540)$
5	$3 \times 7 \times 11$	$2 \times 2 \times 5$	$2 \times 2 \times 3 \times 7 \times 11$	$(20, 924)$
7	$3 \times 5 \times 11$	$2 \times 2 \times 7$	$2 \times 2 \times 3 \times 5 \times 11$	$(28, 660)$
11	$3 \times 5 \times 7$	$2 \times 2 \times 11$	$2 \times 2 \times 3 \times 5 \times 7$	$(44, 420)$
3×5	7×11	$2 \times 2 \times 3 \times 5$	$2 \times 2 \times 7 \times 11$	$(60, 308)$
3×7	5×11	$2 \times 2 \times 3 \times 7$	$2 \times 2 \times 5 \times 11$	$(84, 220)$
3×11	5×7	$2 \times 2 \times 3 \times 11$	$2 \times 2 \times 5 \times 7$	$(132, 140)$

There are 8 different pairs of positive whole numbers having a greatest common factor of 4 and a lowest common multiple of 4620.

ANSWER: (D)

25. Since $12 \times 12 \times 12 = 1728$, Jonas uses each of his 1728 copies of a $1 \times 1 \times 1$ cube with the net shown to build the large cube.
The net contains the numbers 100 and c only, and so each of the numbers appearing on the exterior faces of the large cube is 100 or c.
Jonas builds the large cube in such a way that the sum of the numbers on the exterior faces is as large as possible.
Since $c < 100$, Jonas builds the large cube so that the number of 100s appearing on the exterior faces is as large as possible (and the number of c's appearing on the exterior faces is as small as possible).
The $1 \times 1 \times 1$ cubes which contribute to the numbers on the exterior faces of the large cube can be classified as one of three types.
We call these three types: corner, edge and inside. In the portion of the large $12 \times 12 \times 12$ cube shown in the diagram below, each of these three types is shown.

(i) A *corner cube* is shown in Figure 1. These are cubes that appear in one of the "corners" of the large cube and so there are 8 such corner cubes.

(ii) An *edge cube* is shown in Figure 2. These are cubes that appear along the edges but not in the corners of the large cube. A cube has 12 edges and each edge of the large cube contains 10 edge cubes, and so there are $10 \times 12 = 120$ such cubes.

(iii) An *inside cube* is shown in Figure 3. These are the remaining cubes that contribute to the numbers on the exterior faces of the large cube. A cube has 6 faces and each face of the large cube contains 10×10 inside cubes, and so there are $6 \times 10 \times 10 = 600$ such cubes.

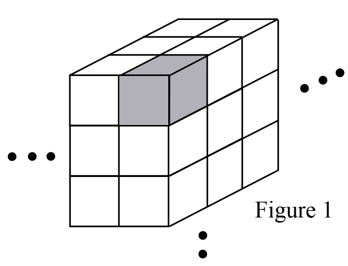


Figure 1

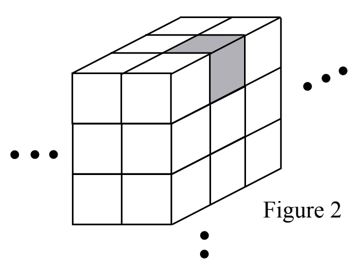


Figure 2

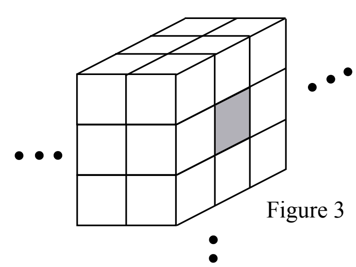


Figure 3

Let S be the sum of the numbers on the exterior faces of the large cube.
Each corner cube has 3 faces which contribute to S. For S to be as large as possible, 100 will

appear on exactly 1 of these 3 faces (there is exactly one 100 in the net of the $1 \times 1 \times 1$ cube), and c will appear on the remaining 2 faces.
Thus, the 8 corner cubes contribute $8 \times 100 + 8 \times 2 \times c$ or $800 + 16c$ to S.

Each edge cube has 2 faces which contribute to S.
For S to be as large as possible, 100 will appear on exactly 1 of these 2 faces and c will appear on the other face.
Thus, the 120 edge cubes contribute $120 \times 100 + 120 \times c$ or $12\,000 + 120c$ to S.

Finally, each inside cube has 1 face which contributes to S.
For S to be as large as possible, 100 will appear on this face.
Thus, the 600 inside cubes contribute 600×100 or $60\,000$ to S.

In total, we get $S = 800 + 16c + 12\,000 + 120c + 60\,000$ and so $S = 136c + 72\,800$.
Since we want S to be at least $80\,000$ and $80\,000 - 72\,800 = 7200$, then $136c$ is at least 7200.
Because $136 \times 52 = 7072$ and $136 \times 53 = 7208$, it must be the case that c is at least 53.
Since we want S to be at most $85\,000$ and $85\,000 - 72\,800 = 12\,200$, then $136c$ is at most $12\,200$.
Because $136 \times 90 = 12\,240$ and $136 \times 89 = 12\,104$, it must be the case that c is at most 89.
This means that c is a positive integer that is at least 53 and at most 89.
There are $89 - 52 = 37$ such integers. (Think of listing the integers from 1 to 89 and removing the integers from 1 to 52.)

Answer: (C)

Grade 8

1. Since 1000 is 1 more than 999, then $1000 + 1000 = 2000$ is 2 more than $999 + 999$.
Thus, $999 + 999 = 2000 - 2 = 1998$.

Answer: (C)

2. An equilateral triangle has 3 sides of equal length.
If the perimeter of an equilateral triangle is 15 m, then the length of each side is $\frac{15 \text{ m}}{3} = 5$ m.

Answer: (B)

3. Since $25 \times 4 = 100$, then 100 is a multiple of 4.
Therefore, the greatest multiple of 4 less than 100 is $24 \times 4 = 96$ (or alternately, $100 - 4 = 96$).

Answer: (B)

4. Points which lie to the right of the y-axis have x-coordinates which are positive.
Points which lie below the x-axis have y-coordinates which are negative.
Point $P(x, y)$ lies to the right of the y-axis and below the x-axis and thus the value of x is positive and the value of y is negative.

Answer: (B)

5. Substituting $x = -6$ into each expression and evaluating, we get

(A) $\begin{aligned} 2 + x &= 2 + (-6) \\ &= -4 \end{aligned}$ (B) $\begin{aligned} 2 - x &= 2 - (-6) \\ &= 2 + 6 \\ &= 8 \end{aligned}$ (C) $\begin{aligned} x - 1 &= -6 - 1 \\ &= -7 \end{aligned}$

(D) $x = -6$ (E) $\begin{aligned} x \div 2 &= (-6) \div 2 \\ &= -3 \end{aligned}$

Of these, $2 - x$ gives the greatest value when $x = -6$.

Answer: (B)

6. At this rate, it would take 6 seconds to fill a 500 mL bottle.
A 250 mL bottle has half the volume of a 500 mL bottle and so it will take half as long or 3 seconds to fill.

Answer: (C)

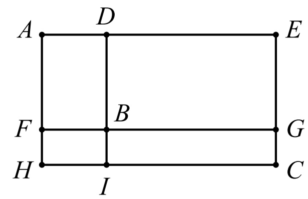
7. If the tens digit of a two-digit number is even, then when the digits are reversed the new number will have a units digit that is even and therefore the number will be even.
If a two-digit number is even, then it is divisible by 2 and so it cannot be a prime number.
Since 29, 23 and 41 each have a tens digit that is even, we may eliminate these three as possible answers.
When the digits of 53 are reversed, the result is 35.
Since 35 is divisible by 5, it is not a prime number.
Finally, when the digits of 13 are reversed, the result is 31.
Since 31 has no positive divisors other than 1 and 31, it is a prime number.

Answer: (D)

8. When 3 red beans are added to the bag, the number of red beans in the bag is $5+3=8$.
When 3 black beans are added to the bag, the number of black beans in the bag is $9+3=12$.
The number of beans now in the bag is $8+12=20$.
If one bean is randomly chosen from the bag, the probability that the bean is red is $\frac{8}{20}=\frac{2}{5}$.

ANSWER: (B)

9. We begin by labelling additional points, as shown.
Beginning at A, the ant may travel right (to D) or down (to F).



Assume the ant begins by travelling right, to D. From D, if the ant continues travelling right (to E) the path cannot pass through B (since the ant can travel only right or down).
From D, the ant must travel down to B. From B, there are two paths that end at C, one travelling right to G and then down to C, and another travelling down to I and then right to C.
Therefore, there are two paths in which the ant begins by travelling right: $A-D-B-G-C$ and $A-D-B-I-C$.

Assume the ant begins by travelling down, to F. From F, if the ant continues travelling down (to H) the path cannot pass through B (since the ant can travel only right or down).
Thus, from F, the ant must travel right to B. From B, there are two paths that end at C, as discussed above.
Therefore, there are two paths in which the ant begins by travelling down: $A-F-B-G-C$ and $A-F-B-I-C$.

There are 4 different paths from A to C that pass through B.

ANSWER: (C)

10. By assigning the largest digits to the largest place values, we form the largest possible four-digit number.
The largest four-digit number that can be formed by rearranging the digits of 2021 is 2210.
By assigning the smallest digits to the largest place values, we form the smallest possible four-digit number.
The smallest four-digit number (greater than 1000) that can be formed by rearranging the digits of 2021 is 1022.
Thus the largest possible difference between two such four-digit numbers is $2210-1022=1188$.

ANSWER: (A)

11. *Solution 1*
PQ and RS intersect at T. Thus, $\angle PTR$ and $\angle STQ$ are vertically opposite angles and so $\angle PTR=\angle STQ=140^\circ$.
Since $\angle PTR=\angle PTU+\angle RTU$, then

$$\begin{aligned}\angle RTU &= \angle PTR-\angle PTU\\ &= 140^\circ-90^\circ\\ &= 50^\circ\end{aligned}$$

and so the measure of $\angle RTU$ is 50°.

Solution 2
RS is a line segment and so $\angle RTQ+\angle STQ=180^\circ$ or $\angle RTQ=180^\circ-140^\circ=40^\circ$.
PQ is a line segment and so $\angle RTQ+\angle RTU+\angle PTU=180^\circ$ or $\angle RTU=180^\circ-40^\circ-90^\circ=50^\circ$.

ANSWER: (C)

12. Given three consecutive integers, the smallest integer is one less than the middle integer and the largest integer is one more than the middle integer.
So, the sum of the smallest and largest integers is twice the middle integer.
Then, the sum of three consecutive integers is equal to 3 times the middle integer and so the sum of three consecutive integers is a multiple of 3.
Of the answers given, the only number that is a multiple of 3 is 21.
Alternately, we could use trial and error to solve this problem to find that $6 + 7 + 8 = 21$.
Answer: (D)

13. Reading from the bar graph, there are 8 yellow shirts, 4 red shirts, 2 blue shirts, and 2 green shirts. In total, the number of shirts is $8 + 4 + 2 + 2 = 16$.
Thus, 8 yellow shirts represents $\frac{8}{16}$ or $\frac{1}{2}$ of the total number of shirts.
The only circle graph showing that approximately half of the shirts are yellow is (E) and thus it probably best represents the information in the bar graph.
We may confirm that this circle graph also shows that approximately $\frac{4}{16} = \frac{1}{4}$ of the shirts are red, approximately $\frac{2}{16} = \frac{1}{8}$ of the shirts are green and approximately $\frac{1}{8}$ of the shirts are blue.
Answer: (E)

14. Let the unknown whole number be n.
Since 16 is a factor of n, then each positive factor of 16 is also a factor of n.
That is, the positive factors of n include 1, 2, 4, 8, 16, and n.
Since 16 is a factor of n, then n is a positive multiple of 16.
The smallest whole number multiple of 16 is 16.
However, if $n = 16$, then n has exactly 5 positive factors, namely 1, 2, 4, 8, and 16.
The next smallest whole number multiple of 16 is 32.
If $n = 32$, then n has exactly 6 positive factors, namely 1, 2, 4, 8, 16, and 32, as required.
Answer: (B)

15. *Solution 1*
The sum of the three interior angles in a triangle is 180°.
A triangle's three interior angles are in the ratio $1 : 4 : 7$ and so the smallest of the angles measures $\frac{1}{1+4+7} = \frac{1}{12}$ of the sum of the three interior angles.
Thus, the smallest angle in the triangle meausres $\frac{1}{12}$ of 180° or $\frac{180^\circ}{12} = 15^\circ$.
The measure of the next largest angle is 4 times the measure of the smallest angle or $4 \times 15^\circ = 60^\circ$.
The measure of the largest angle is 7 times the measure of the smallest angle or $7 \times 15^\circ = 105^\circ$.
The measures of the interior angles are 15°, 60° and 105°.

Solution 2
The sum of the three interior angles in a triangle is 180°.
Working backward from the possible answers, we may eliminate (B) and (C) since the sum of the three given angles is not 180°.
The measures of the smallest and largest angles in the triangle are in the ratio $1 : 7$.
Since $7 \times 12^\circ = 84^\circ$ and not 120°, we may eliminate (A).
Since $7 \times 14^\circ = 98^\circ$ and not 110°, we may eliminate (E).
The remaining possibility is (D) and we may confirm that 15°, 60° and 105° have a sum of 180° and are in the ratio $1 : 4 : 7$.
Answer: (D)

16. The seven numbers $1, 2, 5, 10, 25, 50, 100$ repeat to form the pattern shown.
Thus, the 7^{th} number in the pattern is 100, the 14^{th} number in the pattern is 100, the 21^{st} number in the pattern is 100, and so on.
The 14^{th} number in the pattern is 100 and so the 15^{th} number in the pattern is 1, the 16^{th} is 2, the 17^{th} is 5, and the 18^{th} number in the pattern is 10.
Since 70 is a multiple of 7, the 70^{th} number in the pattern is also 100 and so the 71^{st} number in the pattern is 1, the 72^{nd} is 2, the 73^{rd} is 5, the 74^{th} is 10, and the 75^{th} number in the pattern is 25.
The sum of the 18^{th} and 75^{th} numbers in the pattern is $10 + 25 = 35$.
Answer: (E)

17. *Solution 1*
Gaussville's soccer team won 40% of their first 40 games.
Thus they won $0.40 \times 40 = 16$ games.
After winning the next n games in a row, they had won $16 + n$ games and had played $40 + n$ games.
At this point, they had won 50% or $\frac{1}{2}$ of their games.
This means that the number of games won, $16+n$, was $\frac{1}{2}$ of the number of games played, $40+n$.
For which of the possible answers is $16 + n$ equal to $\frac{1}{2}$ of $40 + n$?
Substituting each of the five possible answers, we get that $16 + 8 = 24$ is $\frac{1}{2}$ of $40 + 8 = 48$, and so the value of n is 8.

Solution 2
Gaussville's soccer team won 40% of their first 40 games.
Thus they had $0.40 \times 40 = 16$ wins and $40 - 16 = 24$ non-wins (losses or ties) in their first 40 games.
At this point, the team went on a winning streak which means they did not accumulate any additional non-wins.
Thus, their 24 non-wins represent 50% of the final total, and so the final number of wins is 24.
Therefore, Gaussville's soccer team won $n = 24 - 16 = 8$ games in a row.
Answer: (D)

18. The fraction of the area of the larger circle that is not shaded does not depend on the actual radius of either circle, and so we begin by letting the radius of the smaller circle be 1 and thus the radius of the larger circle is 3.
In this case, the area of the smaller circle is $\pi(1)^2 = \pi$.
The area of the larger circle is $\pi(3)^2 = 9\pi$.
The area of the larger circle that is not shaded is $9\pi - \pi = 8\pi$.
Therefore, the fraction of the area of the larger circle that is not shaded is $\frac{8\pi}{9\pi} = \frac{8}{9}$.
(Alternately, we could note that the fraction of the area of the larger circle that is shaded is $\frac{\pi}{9\pi} = \frac{1}{9}$ and so the fraction of the area of the larger circle that is not shaded is $1 - \frac{1}{9} = \frac{8}{9}$.)
Answer: (A)

19. We proceed to work backward from the final sum, 440, 'undoing' each of the three operations to determine the sum of their two numbers before any operations were performed.
The final operation performed by each of Asima and Nile was to multiply their number by 4.
Multiplying each of their numbers by 4 increases the sum of the two numbers by a factor of 4.
That is, the final sum of their two numbers was 440, and so the sum of their two numbers immediately before the last operation was performed was $440 \div 4 = 110$.
The second operation performed by each of Asima and Nile was to subtract 10 from their

number.
Subtracting 10 from each of their numbers decreases the sum of the two numbers by 20.
That is, the sum of their two numbers immediately following the second operation was 110, and so the sum of their two numbers immediately before the second operation was performed was $110 + 20 = 130$.
Finally, the first operation performed by each of Asima and Nile was to double their number.
Doubling each of their numbers increases the sum of the two numbers by a factor of 2.
That is, the sum of their two numbers immediately following the first operation was 130, and so the sum of their two numbers before the first operation was performed was $130 \div 2 = 65$.
Each of their original integers is greater than 0 and the two integers have a sum of 65.
Therefore, Asima's original integer could be any integer from 1 to 64, inclusive.
Thus, there are 64 possibilities for Asima's original integer.

ANSWER: (A)

20. *Solution 1*
The table below shows the possible differences between the number on Ruby's roll and the number on Sam's roll.

Number on Ruby's Roll

Number on Sam's Roll	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

Looking at the table above, the total number of possible outcomes is $6 \times 6 = 36$.
The total number of outcomes for which the difference is negative is 15.
The probability that the result from subtracting the number on Sam's roll from the number on Ruby's roll is negative is $\frac{15}{36} = \frac{5}{12}$.

Solution 2
Ruby and Sam each have 6 possible outcomes when they roll the dice, and so the total number of possible outcomes is $6 \times 6 = 36$.
Of these 36 possible outcomes, there are 6 outcomes in which Sam and Ruby each roll the same number and thus the difference between the numbers rolled is 0.
For the remaining $36 - 6 = 30$ possible outcomes, it is equally probable that the number on Ruby's roll is greater than the number on Sam's roll as it is that the number on Sam's roll is greater than the number on Ruby's roll.
That is, one half of these 30, or 15 possible outcomes have a result that is negative and 15 have a result that is positive.

Therefore, the probability that the result from subtracting the number on Sam's roll from the number on Ruby's roll is negative is $\frac{15}{36} = \frac{5}{12}$.

ANSWER: (B)

21. If n is a positive integer, 10^n is equal to 1 followed by n zeros, when evaluated.
For example, $10^4 = 10\,000$ and 10^{2021} is equal to $1\underbrace{00...0}_{\text{2021 0s}}$.

The result of adding 1 to the positive integer consisting of only n 9s is 10^n.
For example, $1 + 9999 = 10\,000 = 10^4$ and $1 + \underbrace{99...9}_{\text{2021 9s}} = 1\underbrace{00...0}_{\text{2021 0s}} = 10^{2021}$.

Let S be the integer equal to $10^{2021} - 2021$.
Since $10^{2021} = 1 + \underbrace{99...9}_{\text{2021 9s}}$, then $S = 1 + \underbrace{99...9}_{\text{2021 9s}} - 2021 = \underbrace{99...9}_{\text{2021 9s}} - 2020 = \underbrace{99...9}_{\text{2017 9s}}7979$.

The sum of the digits of the integer equal to $10^{2021} - 2021$ is $2019 \times 9 + 2 \times 7 = 18\,185$.

ANSWER: (E)

22. We begin by listing the prime numbers up to and including 31.
We choose to end the list at 31 since $30 \times 30 = 900$ and so 31×37 is greater than 900.
This list of prime numbers is

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$$

Considering consecutive pairs of prime numbers from the list above, we get 10 positive integers less than 900 that can be written as a product of two consecutive prime numbers. These are

$$2 \times 3 = 6 \qquad 3 \times 5 = 15 \qquad 5 \times 7 = 35 \qquad 7 \times 11 = 77 \qquad 11 \times 13 = 143$$

$$13 \times 17 = 221 \qquad 17 \times 19 = 323 \qquad 19 \times 23 = 437 \qquad 23 \times 29 = 667 \qquad 29 \times 31 = 899$$

There are 3 positive integers less than 900 that can be written as a product of three consecutive prime numbers. These are

$$2 \times 3 \times 5 = 30 \qquad 3 \times 5 \times 7 = 105 \qquad 5 \times 7 \times 11 = 385$$

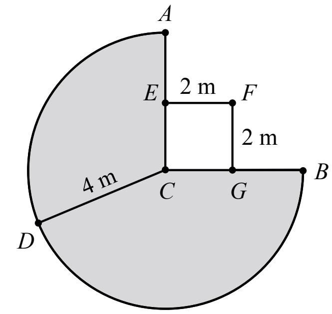
Since each positive integer greater than 1 is either a prime number or can be written as a unique product of prime numbers, it must be that these 3 numbers, 30, 105, 385, are different than those obtained from a product of two consecutive prime numbers. (Alternatively, we could check that 30, 105 and 385 do not appear in the previous list.)
Further, we note that the next smallest positive integer that can be written as a product of three consecutive prime numbers is $7 \times 11 \times 13 = 1001$ which is greater than 900.
Exactly one positive integer less than 900 can be written as a product of four consecutive prime numbers. This number is

$$2 \times 3 \times 5 \times 7 = 210$$

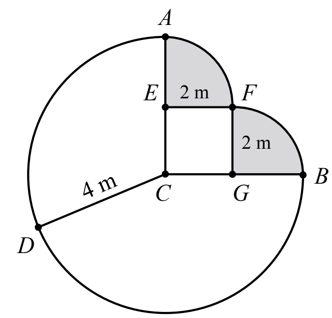
The next smallest positive integer that can be written as a product of four consecutive prime numbers is $3 \times 5 \times 7 \times 11 = 1155$ which is greater than 900.
There are no positive integers that can be written as a product of five or more consecutive prime numbers since $2 \times 3 \times 5 \times 7 \times 11 = 2310$ which is greater than 900.
In total, the number of positive integers less than 900 that can be written as a product of two or more consecutive prime numbers is $10 + 3 + 1 = 14$.

ANSWER: (A)

23. We begin by labelling some additional points as shown.
With the leash extended the full 4 m, the dog can reach points A, B and D where $AC = BC = DC = 4$ m and thus the dog is able to play anywhere inside the shaded area shown.
The doghouse $EFGC$ is a square and so $\angle ECG = 90°$.
Therefore, the shaded figure is $\frac{3}{4}$ of a circle centred at C, has radius $CD = 4$ m, and thus has area $\frac{3}{4}\pi(4\text{ m})^2 = 12\pi\text{ m}^2$.



There is additional area outside the doghouse in which the dog can play, as shaded in the diagram.
Since $AC = 4$ m and $EC = 2$ m, then $AE = 2$ m. Since $EC + EF = 2\text{ m} + 2\text{ m} = 4$ m, the length of the leash, then the dog can reach F.
Similarly, $BG = 2$ m and $GC + GF = 2\text{ m} + 2\text{ m} = 4$ m, the length of the leash, and so the dog can also reach F by travelling around sides GC and GF of the doghouse.
Since $\angle AEF = \angle FGB = 90°$, then each of these two shaded figures is $\frac{1}{4}$ of a circle (centred at E and G respectively), has radius 2 m, and thus each has area $\frac{1}{4}\pi(2)^2 = \pi\text{ m}^2$. (For example, when the dog is above and to the right of E, it can stretch to at most 2 m of rope and so can form a quarter circle of radius 2 m coming from E.)



The area outside of the doghouse in which the dog can play is $12\pi\text{ m}^2 + 2 \times \pi\text{ m}^2 = 14\pi\text{ m}^2$.

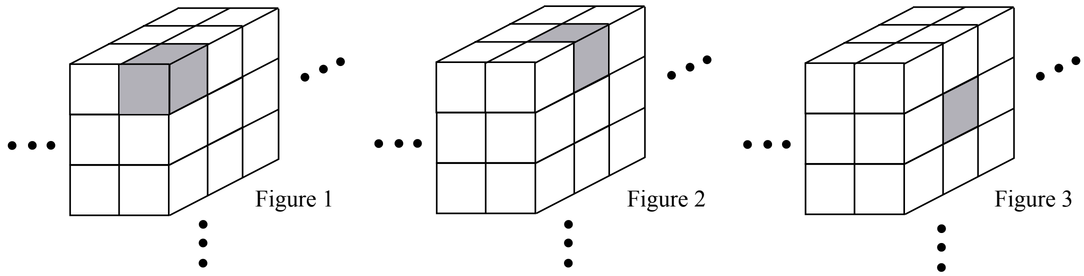
Answer: (A)

24. Let the sum of the numbers on the exterior faces of the $n \times n \times n$ cube be S.
To determine the smallest value of n for which $S > 1500$, we choose to position the $1 \times 1 \times 1$ cubes within the large cube in such a way that S is as large as possible.
The $1 \times 1 \times 1$ cubes which contribute to the numbers on the exterior faces of the large cube can be classified as one of three types.
We call these three types: corner, edge and inside.
In the portion of the large $n \times n \times n$ cube shown in the diagram below, each of these three types is shown.

(i) A *corner cube* is shown in Figure 1. These are cubes that appear in one of the "corners" of the large cube and so there are 8 such corner cubes.

(ii) An *edge cube* is shown in Figure 2. These are cubes that appear along the edges but not in the corners of the large cube.
A cube has 12 edges and each edge of the large cube contains $n - 2$ edge cubes, and so there are $12 \times (n - 2)$ such cubes.

(iii) An *inside cube* is shown in Figure 3. These are the remaining cubes that contribute to the numbers on the exterior faces of the large cube.
A cube has 6 faces and each face of the large cube contains $(n-2) \times (n-2)$ inside cubes, and so there are $6 \times (n - 2)^2$ such cubes.



Each corner cube has 3 faces which contribute to S.
For S to be as large as possible, the $1 \times 1 \times 1$ cube must be positioned in such a way that the sum of the 3 external faces is as large as possible.
We may determine from the given net that the three faces meeting at a vertex of the $1 \times 1 \times 1$ cube may contain the numbers: $-1, 0, 1$ or $-1, 2, 0$ or $-2, 2, 0$ or $-2, 1, 0$.
The sums of these three faces are $0, 1, 0$, and -1 respectively.
To make S as large as possible, we choose to place each corner cube so that the numbers appearing on the exterior faces of the large cube are $-1, 2$ and 0, and thus have the largest possible sum of 1.
Therefore, the 8 corner cubes contribute $8 \times 1 = 8$ to S.

Each edge cube has 2 faces which contribute to S. For S to be as large as possible, the $1 \times 1 \times 1$ cube must be positioned in such a way that the sum of the 2 external faces is as large as possible.
We may determine from the given net that two faces which share an edge of the $1 \times 1 \times 1$ cube contain the numbers: $-1, 0$ or $-1, 2$ or $-1, 1$ or $2, 0$ or $1, 0$ or $-2, 0$ or $-2, 1$ or $-2, 2$.
The sums of these two adjacent faces are $-1, 1, 0, 2, 1, -2, -1$, and 0 respectively.
To make S as large as possible, we choose to place each edge cube so that the numbers appearing on the exterior faces of the large cube are 2 and 0, and thus have the largest possible sum of 2.
Therefore, the $12 \times (n-2)$ edge cubes contribute $2 \times 12 \times (n-2)$ or $24 \times (n-2)$ to S.

Finally, each inside cube has 1 face which contributes to S. For S to be as large as possible, we position each of these cubes so that 2 will appear on this face (since 2 is the largest number in the given net).
Thus, the $6 \times (n-2)^2$ inside cubes contribute $2 \times 6 \times (n-2)^2$ or $12 \times (n-2)^2$ to S.

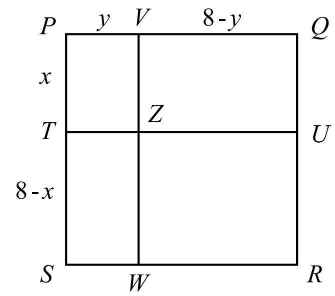
In total, we get $S = 8 + 24 \times (n-2) + 12 \times (n-2)^2$.
We want the smallest value of n for which $S > 1500$ or $8 + 24 \times (n-2) + 12 \times (n-2)^2 > 1500$.
Using trial and error with the given answers, we get the following:

- When $n = 9$, $S = 8 + 24 \times (n-2) + 12 \times (n-2)^2 = 8 + 24 \times 7 + 12 \times 7^2 = 764$ which is less than 1500.
- When $n = 11$, $S = 8 + 24 \times 9 + 12 \times 9^2 = 1196$ which is less than 1500.
- When $n = 12$, $S = 8 + 24 \times 10 + 12 \times 10^2 = 1448$ which is less than 1500.
- When $n = 13$, $S = 8 + 24 \times 11 + 12 \times 11^2 = 1724$ which is greater than 1500.

The smallest value of n for which the sum of the exterior faces of the $n \times n \times n$ cube is greater than 1500 is 13.

ANSWER: (D)

25. We begin by drawing a well-labelled diagram, as shown.



Since $PS = 8$, by letting $PT = x$ we get $TS = 8 - x$. Similarly if $PV = y$, then $VQ = 8 - y$.
The value of x, where $0 < x < 8$, uniquely determines the position of line segment TU (which is parallel to PQ).
The value of y, where $0 < y < 8$, uniquely determines the position of line segment VW (which is parallel to QR).
Thus, the value of N is equal to the number of ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.

Since the areas of rectangles $PVZT$ and $VQUZ$ are integers, then the area of rectangle $PQUT$ is given by the sum of two integers and thus is also an integer.
The area of $PQUT$ is equal to $(PQ)(PT) = 8x$ and so $8x$ is an integer.
Similarly, the area of $PVWS$ is an integer and so $8y$ is an integer.
For what values of x is $8x$ an integer?
If x is an integer, then $8x$ is an integer and so x can equal any integer from 1 to 7 inclusive.
Similarly, y can equal any integer from 1 to 7 inclusive.
For each of the 7 possible values of x, there are 7 possible values for y, and so in this case where each of x and y is an integer, there are $7 \times 7 = 49$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (2, 3)$.
In this case, $8 - x = 6$, $8 - y = 5$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $2 \times 3 = 6$, $6 \times 3 = 18$, $2 \times 5 = 10$, and $6 \times 5 = 30$, respectively.

Having considered the cases in which both x and y are integers, next we will consider the possible cases in which exactly one of x and y is an integer.
Specifically, we will assume that y is an integer and x is not an integer.

Are there non-integer values of x for which $8x$ is an integer?
When $x = \frac{1}{2}$, $8x = 8 \times \frac{1}{2} = 4$ and thus there exist fractional values of x for which $8x$ is an integer.
Let $x = \frac{a}{b}$ where a and b are positive integers and a and b have no factors in common (that is, $\frac{a}{b}$ is in lowest terms).
Since $8x = 8 \times \frac{a}{b}$ is an integer exactly when b is a positive divisor of 8, then b could equal $1, 2, 4$, or 8.
Next, we consider each of these possible values for b.

When $b = 1$, $x = \frac{a}{b} = \frac{a}{1} = a$ and so x is an integer.
Integer values of both x and y is the case previously considered in which we determined there are 49 ordered pairs (x, y) that satisfy the given conditions.

Next, we consider the cases for which $b = 2$.
Since a and b have no common factors, then a must be odd.
Further, $0 < \frac{a}{b} < 8$ and so a can be $1, 3, 5, 7, 9, 11, 13$, or 15 (note that $\frac{17}{2} > 8$).
When x is equal to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$, or $\frac{15}{2}$, what are the possible values of y?
Recall that we are considering cases in which exactly one of x and y is an integer and thus we require y to be an integer.
The area of rectangle $PVZT$ is an integer and so $(PV)(PT) = xy$ is an integer. (We note that the area of each of the other 3 rectangles is also an integer.)
When x is equal to a fraction of the form $\frac{a}{2}$ (where a is odd and comes from the list above),

and xy is an integer, then 2 is a factor of y and so y is an even integer.
Since $0 < y < 8$, then y can be $2, 4$ or 6.
For each of the 8 choices for x, there are 3 choices for y and so in this case there are $8 \times 3 = 24$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (\frac{3}{2}, 6)$. In this case, $8 - x = \frac{13}{2}$, $8 - y = 2$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $\frac{3}{2} \times 6 = 9$, $\frac{13}{2} \times 6 = 39$, $\frac{3}{2} \times 2 = 3$, and $\frac{13}{2} \times 2 = 13$, respectively.
Recall from earlier that $8y$ is also an integer.
Thus, each of the possible values of x is a possible value for y, and vice versa.
That is, y can be equal to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$, or $\frac{15}{2}$, which gives x equal to 2,4 or 6, and 24 additional ordered pairs (x, y).

To summarize to this point, if both x and y are integers, there are 49 ordered pairs (x, y) which satisfy the given conditions.
If x is a fraction of the form $\frac{a}{2}$ for odd integers $1 \leq a \leq 15$ and y is an integer, there are 24 ordered pairs (x, y) which satisfy the given conditions.
If y is a fraction of the form $\frac{a}{2}$ for odd integers $1 \leq a \leq 15$ and x is an integer, there are 24 ordered pairs (x, y) which satisfy the given conditions.

Next, we consider the cases for which $b = 4$.
Since a and b have no common factors, then a must be odd.
Further, $0 < \frac{a}{b} < 8$ and so a can be $1, 3, 5, \ldots, 27, 29$, or 31 (note that $\frac{33}{4} > 8$).
When x is equal to $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \frac{27}{4}, \frac{29}{4}$, or $\frac{31}{4}$, what are the possible values of y?
Recall that we are considering cases in which exactly one of x and y is an integer and thus we require y to be an integer.
Since xy is an integer and x is equal to a fraction of the form $\frac{a}{4}$, then 4 is a factor of y.
Since $0 < y < 8$, then $y = 4$.
For each of the 16 choices for x, there is 1 choice for y and so in this case there are $16 \times 1 = 16$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (\frac{5}{4}, 4)$.
In this case, $8 - x = \frac{27}{4}$, $8 - y = 4$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $\frac{5}{4} \times 4 = 5$, $\frac{27}{4} \times 4 = 27$, $\frac{5}{4} \times 4 = 5$, and $\frac{27}{4} \times 4 = 27$, respectively.
When y is equal to $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \frac{27}{4}, \frac{29}{4}$, or $\frac{31}{4}$, and $x = 4$, there are 16 additional ordered pairs (x, y).

To complete the cases for which exactly one of x and y is an integer, we consider $b = 8$.
Since xy is an integer and x is equal to a fraction of the form $\frac{a}{8}$, then 8 is a factor of y.
However, $y < 8$ and thus cannot have a factor of 8 and so there are no ordered pairs (x, y) when $b = 8$.

Finally, we consider cases in which both x and y are not integers.
As previously determined, if x is not an integer, then it is of the form $\frac{a}{2}$ or $\frac{a}{4}$ for some positive odd integers a.
If y is not an integer, then it is similarly of the form $\frac{a}{2}$ or $\frac{a}{4}$ for some positive odd integers a.
However, if x and y are each of this form, then their product xy is of the form $\frac{c}{4}$, $\frac{c}{8}$ or $\frac{c}{16}$ where c is the product of two odd integers and thus is odd.
Since it is not possible for 4, 8 or 16 to be a factor of an odd number, then xy cannot be an integer and so the area of rectangle $PVZT$ cannot be an integer in this case.
Thus, there are no cases for which both x and y are not integers.

Therefore, the value of N is equal to $49 + (2 \times 24) + (2 \times 16) = 49 + 48 + 32 = 129$, and so the remainder when $N^2 = 16\,641$ is divided by 100 is 41.

ANSWER: (D)