The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Galois Contest

April 2021
(in North America and South America)

April 2021
(outside of North America and South America)

Solutions

1. (a) Substituting $a = 5$ and $b = 1$, we get $5\triangle 1 = 5(2 \times 1 + 4) = 5(6) = 30$.

   (b) If $k\triangle 2 = 24$, then $k(2 \times 2 + 4) = 24$ or $8k = 24$, and so $k = 3$.

   (c) Solving the given equation for p, we get

$$\begin{aligned} p\triangle 3 &= 3\triangle p \\ p(2 \times 3 + 4) &= 3(2p + 4) \\ p(10) &= 6p + 12 \\ 10p - 6p &= 12 \\ 4p &= 12 \\ p &= 3 \end{aligned}$$

   The only value of p for which $p\triangle 3 = 3\triangle p$ is $p = 3$.

   (d) Simplifying the given equation, we get

$$\begin{aligned} m\triangle(m + 1) &= 0 \\ m(2(m + 1) + 4) &= 0 \\ m(2m + 2 + 4) &= 0 \\ m(2m + 6) &= 0 \end{aligned}$$

   Thus, $m = 0$ or $2m + 6 = 0$ which gives $m = -3$.
   The values of m for which $m\triangle(m + 1) = 0$ are $m = 0$ and $m = -3$.
   (Substituting each of these values of m, we may check that $0\triangle 1 = 0(2\times 1+4) = 0(6) = 0$, and that $(-3)\triangle(-2) = -3(2 \times (-2) + 4) = -3(0) = 0$.)

2. (a) Team P played 27 games which included 10 wins and 14 losses.
   Thus, Team P had $27 - 10 - 14 = 3$ ties at the end of the season.

   (b) Team Q had 2 more wins than Team P, or $10 + 2 = 12$ wins.
   Team Q had 4 fewer losses than Team P, or $14 - 4 = 10$ losses.
   Since Team Q played 27 games, they had $27 - 12 - 10 = 5$ ties.
   At the end of the season, Team Q had a total of $(2 \times 12) + (0 \times 10) + (1 \times 5)$ or 29 points.

   (c) *Solution 1*
   Assume that Team R finished the season with exactly 6 ties.
   Since 6 ties contribute 6 points to their points total, then Team R earned the remaining $25 - 6 = 19$ points as a result of their wins.
   However, each win contributes 2 points to the total, and thus it is not possible to earn an odd number of points from wins.
   Therefore, Team R could not have finished the season with exactly 6 ties.

   Solution 2
   Assume that Team R finished the season with exactly w wins.
   If Team R finished with exactly 6 ties, then they finished the season with a total of $(2 \times w) + (1 \times 6)$ or $2w + 6 = 2(w + 3)$ points (they earn 0 points for losses).
   Since w is an integer, then $w + 3$ is an integer and so $2(w + 3)$ is an even integer.
   However, this is not possible since Team R finished the season with 25 points, an odd number of points.
   Therefore, Team R could not have finished the season with exactly 6 ties.

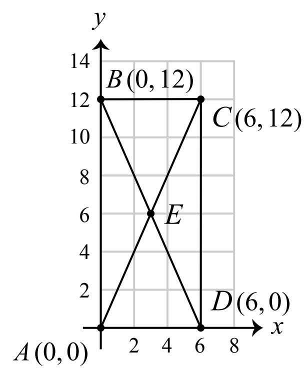
(d) *Solution 1*

Let the number of losses that Team S had at the end of the season be ℓ.
Team S had 4 more wins than losses and thus finished the season with $\ell + 4$ wins.
Since Team S played 27 games, then each of their remaining $27 - \ell - (\ell + 4) = 23 - 2\ell$ games resulted in a tie.
Therefore, Team S finished the season with a total of $(2 \times (\ell + 4)) + (0 \times \ell) + (1 \times (23 - 2\ell))$ or $2\ell + 8 + 23 - 2\ell = 31$ points.

Solution 2

Each of the 4 teams played 27 games, 2 teams played in each game, and so the season finished with a total of $\frac{4\times 27}{2} = 54$ games played.
Each of the 54 games resulted in a total of 2 points being awarded (either 2 points to a winning team and 0 to the losing team or 1 point to each of the two teams that tied).
Thus, the total points earned by all 4 teams at the end of the season was $2 \times 54 = 108$.
The table shows that Team P finished with 23 points, Team R had 25 points, and in part (b) we determined that Team Q had 29 points at the end of the season.
Therefore, Team S finished the season with $108 - 23 - 25 - 29 = 31$ points.

3. (a) *Solution 1*

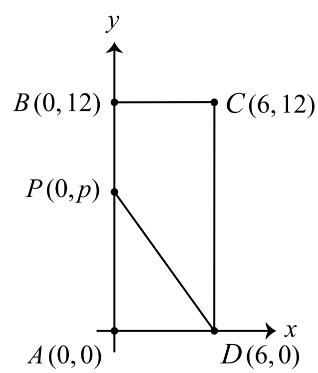


We begin by drawing and labelling a diagram, as shown.
The diagonals of a rectangle intersect at the centre of the rectangle. That is, E is the midpoint of AC. Thus, the x-coordinate of E is the average of the x-coordinates of A and C, or $\frac{0+6}{2} = 3$.
The y-coordinate of E is the average of the y-coordinates of A and C, or $\frac{0+12}{2} = 6$, and so the coordinates of E are $(3, 6)$.
Consider base $AD = 6$ of $\triangle ADE$, then its height is equal to the distance from E to the x-axis, which is 6.
The area of $\triangle ADE$ is $\frac{1}{2}(6)(6) = 18$.

Solution 2

The diagonals of a rectangle divide the rectangle into 4 non-overlapping triangles having equal area. (You should consider why this is true before reading on.)
Thus, the area of $\triangle ADE$ is equal to $\frac{1}{4}$ of the area of rectangle $ABCD$ or $\frac{1}{4}(6)(12) = 18$.

(b) *Solution 1*



We begin by drawing and labelling a diagram, as shown.
The area of rectangle $ABCD$ is equal to the area of trapezoid $BCDP$ plus the area of $\triangle PAD$.
Since the area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, then the area of $\triangle PAD$ is $\frac{1}{3}$ the area of $ABCD$ (and the area of trapezoid $BCDP$ is $\frac{2}{3}$ the area of $ABCD$).
The area of rectangle $ABCD$ is $6 \times 12 = 72$, and so the area of $\triangle PAD$ is $\frac{1}{3} \times 72 = 24$.
The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$, and so $3p = 24$ or $p = 8$.

Solution 2

Point P has coordinates $(0, p)$ and so $AP = p$ and $BP = 12 - p$.
The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$.
The area of trapezoid $BCDP$ is $\frac{1}{2}(BC)(BP + CD) = \frac{1}{2}(6)(12 - p + 12) = 3(24 - p)$.

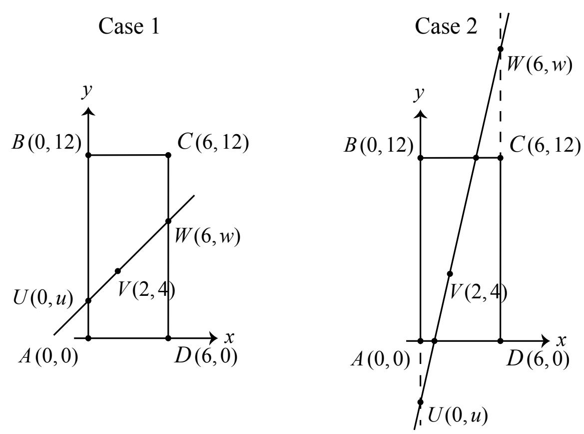
The area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, and so $3(24 - p) = 2(3p)$ or $24 - p = 2p$, and so $3p = 24$ or $p = 8$.

(c) The area of rectangle $ABCD$ is $6 \times 12 = 72$.
The sum of the areas of the two trapezoids is equal to the area of rectangle $ABCD$.
Since the ratio of the areas of these two trapezoids is $5 : 3$, then the areas of the two trapezoids are $\frac{5}{8} \times 72 = 45$ and $\frac{3}{8} \times 72 = 27$.
(We may check that $45 : 27 = 5 : 3$ and $45 + 27 = 72$.)

Let ℓ be the line that passes through U, V and W.
Begin by assuming ℓ does not pass through a vertex of $ABCD$. In this case, ℓ either intersects opposite sides of $ABCD$, or it intersects adjacent sides of $ABCD$.
If ℓ intersects opposite sides of $ABCD$, then ℓ divides $ABCD$ into two trapezoids, as required.
If ℓ intersects adjacent sides of $ABCD$, then ℓ divides $ABCD$ into a triangle and a pentagon. This is not possible.

Assume ℓ passes through at least one vertex of $ABCD$.
In this case, ℓ divides $ABCD$ into two figures, at least one of which is a triangle. This is not also possible.
Thus, ℓ intersects opposite sides of $ABCD$ and does not pass through A, B, C, or D.

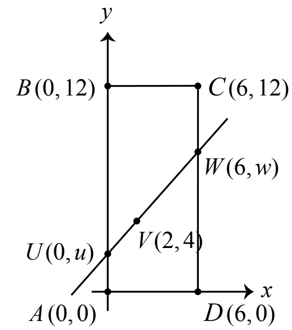
That is, line ℓ can intersect opposite sides of $ABCD$ in the two different ways shown below.



In each case, since ℓ is a straight line passing through U, V and W, then the slope of UV is equal to the slope of VW.
That is,

$$\begin{aligned} \frac{4-u}{2-0} &= \frac{w-4}{6-2} \\ 4(4-u) &= 2(w-4) \\ 2(4-u) &= w-4 \\ 8-2u &= w-4 \\ w &= 12-2u \end{aligned}$$

Case 1: Line ℓ intersects sides AB and CD.
That is, U lies between A and B, and W lies between C and D.



In this case, $0 < u < 12$, $0 < w < 12$, $AU = u$, and $DW = w$.
The area of trapezoid $ADWU$ is

$$\tfrac{1}{2}(AD)(DW + AU) = \tfrac{1}{2}(6)(w + u) = 3(w + u).$$

Since $w = 12 - 2u$, the area of trapezoid $ADWU$ becomes $3(12 - u)$.
We consider each of two possibilities: the area of trapezoid $ADWU$ is equal to 27, or the area is equal to 45.

If the area of trapezoid $ADWU$ is equal to 27, then

$$\begin{aligned} 3(12 - u) &= 27 \\ 12 - u &= 9 \\ u &= 3 \end{aligned}$$

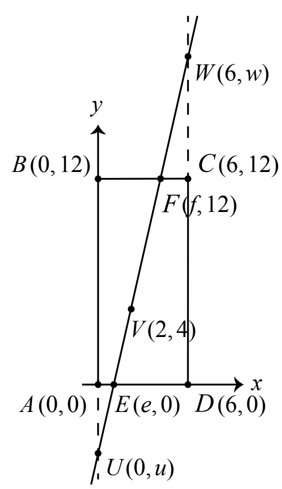
Substituting $u = 3$ into $w = 12 - 2u$, we get $w = 12 - 6 = 6$.
The Case 1 conditions that $0 < u < 12$ and $0 < w < 12$ are satisfied and thus the ratio of the areas of the two trapezoids is $5 : 3$ for the pair of points $U(0, 3)$ and $W(6, 6)$.

If the area of trapezoid $ADWU$ is equal to 45, then

$$\begin{aligned} 3(12 - u) &= 45 \\ 12 - u &= 15 \\ u &= -3 \end{aligned}$$

Here, the condition that $0 < u < 12$ is not satisfied and so there is no pair of points U and W for which the ratio of the areas of the two trapezoids is $5 : 3$.

Case 2: Line ℓ intersects sides AD and BC.
That is, U lies on AB extended, outside of side AB, and W lies on CD extended, outside of side CD.



We begin by drawing and labelling a diagram, including $E(e, 0)$ and $F(f, 12)$, the points where ℓ intersects sides AD and BC respectively, as shown.
In this case, $u < 0$ and $w > 12$ (as in the diagram shown), or $u > 12$ and $w < 0$ (when U lies above B and W lies below D).
We note that what follows is true for each of these two cases, and thus we need not consider them separately.
In this case, we require that $0 < e < 6$, $0 < f < 6$, and so we get $BF = f$ and $AE = e$.
The area of trapezoid $BFEA$ is

$$\tfrac{1}{2}(AB)(BF + AE) = \tfrac{1}{2}(12)(f + e) = 6(f + e).$$

Further, since ℓ is a straight line passing through E, V and F, then the slope of EV is equal to the slope of FV.

That is,

$$\begin{aligned}\frac{4-0}{2-e} &= \frac{12-4}{f-2}\\ \frac{4}{2-e} &= \frac{8}{f-2}\\ 4(f-2) &= 8(2-e)\\ f-2 &= 2(2-e)\\ f &= 6-2e\end{aligned}$$

Since $f = 6 - 2e$, the area of trapezoid $BFEA$ becomes $6(6 - e)$.
We consider each of two possibilities: the area of trapezoid $BFEA$ is equal to 27, or the area is equal to 45.

If the area of trapezoid $BFEA$ is equal to 27, then

$$\begin{aligned}6(6-e) &= 27\\ 6-e &= \tfrac{9}{2}\\ e &= \tfrac{3}{2}\end{aligned}$$

Substituting $e = \frac{3}{2}$ into $f = 6 - 2e$, we get $f = 3$, and these values satisfy the Case 2 conditions $0 < e < 6$ and $0 < f < 6$.
Here, we get $E(\frac{3}{2}, 0)$ and $F(3, 12)$ and use these points to determine U and W.
The slope of FV is $\dfrac{12-4}{3-2} = 8$ and so the slope of WV is also 8, which gives $\dfrac{w-4}{4} = 8$, and solving we get $w = 36$.
Similarly, the slope of VU is also 8, which gives $\dfrac{4-u}{2} = 8$, and solving we get $u = -12$.
We note that $w = 36$ and $u = -12$ satisfy the conditions $w > 12$ and $u < 0$ and so the ratio of the areas of the two trapezoids is $5 : 3$ for the points $U(0, -12)$ and $W(6, 36)$.

If the area of trapezoid $BFEA$ is equal to 45, then

$$\begin{aligned}6(6-e) &= 45\\ 6-e &= \tfrac{15}{2}\\ e &= -\tfrac{3}{2}\end{aligned}$$

Here, the condition that $0 < e < 6$ is not satisfied and so there is no pair of points E and F and thus no pair of points U and W for which the ratio of the areas of the two trapezoids is $5 : 3$.

Thus, there are two pairs of points U and W for which the ratio of the areas of the two trapezoids is $5 : 3$. These are $U(0, 3)$, $W(6, 6)$, and $U(0, -12)$, $W(6, 36)$.

4. (a) When $x = 6$, $\frac{5}{x} + \frac{14}{y} = 2$ becomes $\frac{5}{6} + \frac{14}{y} = 2$ and so $\frac{14}{y} = 2 - \frac{5}{6}$ or $\frac{14}{y} = \frac{7}{6}$, which gives $y = 12$.

(b) *Solution 1*

Since x and y are positive integers, we obtain the following equivalent equations,

$$\begin{aligned} \frac{4}{x} + \frac{5}{y} &= 1 \\ \frac{4}{x}(xy) + \frac{5}{y}(xy) &= 1(xy) \qquad \text{(since } xy \neq 0) \\ 4y + 5x &= xy \\ xy - 5x - 4y &= 0 \\ x(y-5) - 4y &= 0 \\ x(y-5) - 4y + 20 &= 20 \\ x(y-5) - 4(y-5) &= 20 \\ (x-4)(y-5) &= 20 \end{aligned}$$

Since x and y are positive integers, then $x - 4$ and $y - 5$ are integers and thus are a factor pair of 20.
Since $y > 0$, then $y - 5 > -5$.
The factors of 20 which are greater than -5 are: $-4, -2, -1, 1, 2, 4, 5, 10$, and 20.
If $y - 5$ is equal to -4, then $x - 4 = -5$ (since $(-5)(-4) = 20$), and so $x = -1$.
This is not possible since x is a positive integer.
Similarly, $y - 5$ cannot equal -2 or -1 (since each gives $x < 0$), and so $y - 5$ is a positive factor of 20.
In the table below, we determine the values of x and y corresponding to each of the positive factor pairs of 20.

Factor Pair	$x-4$	$y-5$	x	y
1 and 20	1	20	5	25
20 and 1	20	1	24	6
2 and 10	2	10	6	15
10 and 2	10	2	14	7
4 and 5	4	5	8	10
5 and 4	5	4	9	9

Thus, the ordered pairs of positive integers (x, y) that are solutions to the given equation are $(5, 25)$, $(24, 6)$, $(6, 15)$, $(14, 7)$, $(8, 10)$, and $(9, 9)$.

Solution 2

Since x and y are positive integers, we obtain the following equivalent equations,

$$\begin{aligned}
\frac{4}{x}+\frac{5}{y} &= 1 \\
\frac{4}{x}(xy)+\frac{5}{y}(xy) &= 1(xy) \qquad \text{(since } xy \neq 0\text{)} \\
4y+5x &= xy \\
xy-5x &= 4y \\
x(y-5) &= 4y \\
x &= \frac{4y}{y-5} \quad (y \neq 5) \\
x &= \frac{4y-20+20}{y-5} \\
x &= \frac{4(y-5)+20}{y-5} \\
x &= 4+\frac{20}{y-5}
\end{aligned}$$

Since x and y are positive integers, then $y-5$ is a divisor of 20.

Since $y>0$, then $y-5>-5$.

The divisors of 20 which are greater than -5 are: $-4,-2,-1,1,2,4,5,10$, and 20.

If $y-5$ is equal to -4, then $x=4+\frac{20}{-4}=-1$, which is not possible since x is a positive integer.

Similarly, $y-5$ cannot equal -2 or -1 (since each gives $x<0$), and so $y-5$ is a positive divisor of 20.

In the table below, we determine the values of y and x corresponding to each of the positive divisors of 20.

$y-5$	1	2	4	5	10	20
y	6	7	9	10	15	25
x	24	14	9	8	6	5

Thus, the ordered pairs of positive integers (x,y) that are solutions to the given equation are $(24,6),(14,7),(9,9),(8,10),(6,15)$, and $(5,25)$.

(c) *Solution 1*

Since $x \geq 1$ and $y \geq 1$, then $\dfrac{16}{x} + \dfrac{25}{y} \leq 16 + 25 = 41$, and so $5 \leq p \leq 41$. That is, the possible prime numbers p come from the list $5, 7, 11, 13, 17, 19, 23, 29, 31, 37$, and 41.

Since x and y are positive integers, we obtain the following equivalent equations,

$$\begin{aligned}
\frac{16}{x} + \frac{25}{y} &= p \\
\frac{16}{x}(xy) + \frac{25}{y}(xy) &= p(xy) \qquad \text{(since } xy \neq 0\text{)} \\
16y + 25x &= pxy \\
pxy - 25x - 16y &= 0 \\
p^2xy - 25px - 16py &= 0 \qquad \text{(since } p > 0\text{)} \\
px(py - 25) - 16py &= 0 \\
px(py - 25) - 16py + 400 &= 400 \\
px(py - 25) - 16(py - 25) &= 400 \\
(px - 16)(py - 25) &= 400
\end{aligned}$$

Since p, x and y are positive integers, then $px - 16$ and $py - 25$ are integers and thus are a factor pair of 400.

Since $p \geq 5$ and $x \geq 1$, then $px \geq 5$, and so $px - 16 \geq 5 - 16$ or $px - 16 \geq -11$.

The factors of 400 which are greater than or equal to -11, and are less than 0, are: $-1, -2, -4, -5, -8$, and -10.

If $px - 16 = -1$, then $py - 25 = -400$.

In this case, we get $py = -375$ which is not possible since both p and y are positive.

We can similarly show that $px - 16$ cannot equal $-2, -4, -5, -8$, and -10 (since each gives $py < 0$) and so $px - 16$ is a positive factor of 400 and thus $py - 25$ is also.

In the table below, we determine possible values of p corresponding to each of the positive factor pairs of 400.

Recall from earlier that we only need to consider possible values of p for which $5 \leq p \leq 41$.

$px - 16$	$py - 25$	px	py	New common prime factor of the integers px and py
1	400	17	$425 = 17 \times 25$	17
2	200	18	225	
4	100	$20 = 5 \times 4$	$125 = 5 \times 25$	5
5	80	$21 = 7 \times 3$	$105 = 7 \times 15$	7
8	50	24	75	
10	40	$26 = 13 \times 2$	$65 = 13 \times 5$	13
16	25	32	50	
20	20	36	45	
25	16	41	41	41
40	10	56	35	
50	8	$66 = 11 \times 6$	$33 = 11 \times 3$	11
80	5	96	30	
100	4	$116 = 29 \times 4$	29	29
200	2	216	27	
400	1	416	26	

The values of p for which there is at least one ordered pair of positive integers (x, y) that is a solution to the given equation are $5, 7, 11, 13, 17, 29$, and 41.
We may check, for example, that when $(x, y) = (6, 3)$ we get,

$$\frac{16}{x} + \frac{25}{y} = \frac{16}{6} + \frac{25}{3} = \frac{16}{6} + \frac{50}{6} = \frac{66}{6} = 11$$

as given in the table above.

Solution 2

Since $x \geq 1$ and $y \geq 1$, then $\frac{16}{x} + \frac{25}{y} \leq 16 + 25 = 41$, and so $5 \leq p \leq 41$. That is, the possible prime numbers p come from the list $5, 7, 11, 13, 17, 19, 23, 29, 31, 37$, and 41.

When x is a positive divisor of 16, $\frac{16}{x}$ is a positive integer.

Specifically, when $x = 1, 2, 4, 8, 16$, the values of $\frac{16}{x}$ are $16, 8, 4, 2, 1$, respectively.

Similarly, when y is a positive divisor of 25, $\frac{25}{y}$ is a positive integer.

Specifically, when $y = 1, 5, 25$, the values of $\frac{25}{y}$ are $25, 5, 1$, respectively.

We may use this observation to determine some values of p for which there is at least one ordered pair of positive integers (x, y) that is a solution to the equation.
We summarize these solutions in the table below.

p	x	y	$\frac{16}{x} + \frac{25}{y}$
5	4	25	$\frac{16}{4} + \frac{25}{25} = 4 + 1$
7	8	5	$\frac{16}{8} + \frac{25}{5} = 2 + 5$
13	2	5	$\frac{16}{2} + \frac{25}{5} = 8 + 5$
17	1	25	$\frac{16}{1} + \frac{25}{25} = 16 + 1$
29	4	1	$\frac{16}{4} + \frac{25}{1} = 4 + 25$
41	1	1	$\frac{16}{1} + \frac{25}{1} = 16 + 25$

From our previous list of possible values of p, we have only 11, 19, 23, 31, and 37 remaining to consider.

Since x and y are positive integers, we obtain the following equivalent equations,

$$\begin{aligned}
\frac{16}{x}+\frac{25}{y} &= p \\
\frac{16}{x}(xy)+\frac{25}{y}(xy) &= p(xy) \qquad \text{(since } xy \neq 0) \\
16y+25x &= pxy \\
pxy-25x &= 16y \\
x(py-25) &= 16y \\
x &= \frac{16y}{py-25} \qquad (p \geq 11 \text{ and so no multiple of } p \text{ can equal } 25)
\end{aligned}$$

Since $x > 0$ and $16y > 0$ and $x = \dfrac{16y}{py-25}$, then $py - 25 > 0$ and so $py > 25$.

Further, x is an integer and so $x \geq 1$, which gives $\dfrac{16y}{py-25} \geq 1$.

Simplifying, we get $16y \geq py - 25$ or $py - 16y \leq 25$, and so $y \leq \dfrac{25}{p-16}$ when $p > 16$.

We may use this inequality to determine restrictions on y given each of the remaining possible values of p which are greater than 16, namely $37, 31, 23$, and 19.

For example if $p = 37$, then $y \leq \dfrac{25}{37-16}$ or $y \leq \dfrac{25}{21}$, and so $y = 1$. However, when $p = 37$ and $y = 1$, we get $x = \dfrac{16(1)}{37(1)-25} = \dfrac{16}{12}$ which is not an integer, and thus $p \neq 37$.

We summarize similar work for $p = 31, 23, 19$ in the table below noting that when $y = 1$ and $p = 23$ or $p = 19$ we get $py < 25$ (earlier we showed $py > 25$), and thus we need not consider these two cases.

p	$y \leq \frac{25}{p-16}$	Possible integer values of y	Corresponding values of $x = \frac{16y}{py-25}$
31	$y \leq \frac{25}{31-16} = \frac{25}{15}$	$y = 1$	$x = \frac{16}{6}$
23	$y \leq \frac{25}{23-16} = \frac{25}{7}$	$y = 2, 3$	$x = \frac{32}{21}, \frac{48}{44}$
19	$y \leq \frac{25}{19-16} = \frac{25}{3}$	$y = 2, 3, 4, 5, 6, 7, 8$	$x = \frac{32}{13}, \frac{48}{32}, \frac{64}{51}, \frac{80}{70}, \frac{96}{89}, \frac{112}{108}, \frac{128}{127}$

Since there are no integer values of x, then $p \neq 19, 23, 31, 37$.

The final remaining value to check is $p = 11$.

As noted earlier, $py > 25$ and so when $p = 11$, we get $y > \frac{25}{11}$ or $y \geq 3$ (since y is an integer).

Trying $y = 3$, we get $x = \dfrac{16(3)}{11(3)-25} = \dfrac{48}{8} = 6$ and so when $p = 11$, $(x, y) = (6, 3)$ is a solution to the equation.

Summarizing, the values of p for which there is at least one ordered pair of positive integers (x, y) that is a solution to the equation are $5, 7, 11, 13, 17, 29$, and 41.