The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Fermat Contest

(Grade 11)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

Solutions

1. A rectangle with width 2 cm and length 3 cm has area 2 cm $\times$ 3 cm $= 6$ cm^2.

   Answer: (E)

2. Calculating, $2 + 3 \times 5 + 2 = 2 + 15 + 2 = 19$.

   Answer: (A)

3. Expressed as a fraction, 25% is equivalent to $\frac{1}{4}$.
   Since $\frac{1}{4}$ of 60 is 15, then 25% of 60 is 15.

   Answer: (B)

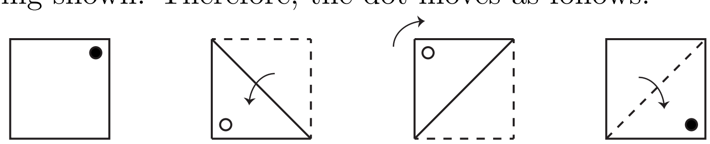
4. When $x \neq 0$, we obtain $\dfrac{4x}{x+2x} = \dfrac{4x}{3x} = \dfrac{4}{3}$.
   Thus, when $x = 2021$, we have $\dfrac{4x}{x+2x} = \dfrac{4}{3}$.
   Alternatively, we could substitute $x = 2021$ to obtain $\dfrac{4x}{x+2x} = \dfrac{8084}{2021+4042} = \dfrac{8084}{6063} = \dfrac{4}{3}$.

   Answer: (B)

5. We note that $6 = 2 \times 3$ and $27 = 3 \times 9$ and $39 = 3 \times 13$ and $77 = 7 \times 11$, which means that each of 6, 27, 39, and 77 can be written as the product of two integers, each greater than 1. Thus, 53 must be the integer that cannot be written in this way. We can check that 53 is indeed a prime number.

   Answer: (C)

6. We draw an unshaded dot to represent the location of the dot when it is on the other side of the sheet of paper being shown. Therefore, the dot moves as follows:

   It is worth noting that folding and unfolding the paper have no net effect on the figure. Thus, the resulting figure can be determined by rotating the original figure by 90° clockwise.

   Answer: (E)

7. When $x = -2$, we get $x^2 = 4$. Here, $x < x^2$.
   When $x = -\frac{1}{2}$, we get $x^2 = \frac{1}{4}$. Here, $x < x^2$.
   When $x = 0$, we get $x^2 = 0$. Here, $x = x^2$.
   When $x = \frac{1}{2}$, we get $x^2 = \frac{1}{4}$. Here, $x > x^2$.
   When $x = 2$, we get $x^2 = 4$. Here, $x < x^2$.
   This means that $x = \frac{1}{2}$ is the only choice where $x > x^2$.

   Answer: (D)

8. Suppose that the original integer has tens digit a and ones (units) digit b.
   This integer is equal to $10a + b$.
   When the digits are reversed, the tens digit of the new integer is b and the ones digit is a.
   This new integer is equal to $10b + a$.
   Since the new two-digit integer minus the original integer is 54, then $(10b + a) - (10a + b) = 54$ and so $9b - 9a = 54$ which gives $b - a = 6$.
   Thus, the positive difference between the two digits of the original integer is 6. An example of a pair of such integers is 71 and 17.

   Answer: (C)

9. The line with equation $y = 2x - 6$ has slope 2. When this line is translated, the slope does not change.
The line with equation $y = 2x - 6$ has y-intercept -6. When this line is translated upwards by 4 units, its y-intercept is translated upwards by 4 units and so becomes -2.
This means that the new line has equation $y = 2x - 2$.
To find its x-intercept, we set $y = 0$ to obtain $0 = 2x - 2$ and so $2x = 2$ or $x = 1$.
Thus, the x-intercept is 1.

ANSWER: (D)

10. Using exponent laws, $3^{x+2} = 3^x \cdot 3^2 = 3^x \cdot 9$.
Since $3^x = 5$, then $3^{x+2} = 3^x \cdot 9 = 5 \cdot 9 = 45$.

ANSWER: (E)

11. Since the second number being added is greater than 300 and the sum has hundreds digit R, then R cannot be 0.
From the ones column, we see that the ones digit of $R + R$ is 0. Since $R \neq 0$, then $R = 5$.
This makes the sum

$$\begin{array}{cccc} & & \overset{1}{} & \\ & P & 7 & 5 \\ + & 3 & 9 & 5 \\ \hline & 5 & Q & 0 \end{array}$$

Since $1 + 7 + 9 = 17$, we get $Q = 7$ and then $1 + P + 3 = 5$ and so $P = 1$, giving the final sum

$$\begin{array}{cccc} & \overset{1}{} & \overset{1}{} & \\ & 1 & 7 & 5 \\ + & 3 & 9 & 5 \\ \hline & 5 & 7 & 0 \end{array}$$

Therefore, $P + Q + R = 1 + 7 + 5 = 13$.

ANSWER: (A)

12. A perfect square is divisible by 9 exactly when its square root is divisible by 3.
In other words, n^2 is divisible by 9 exactly when n is divisible by 3.
In the list $1, 2, 3, \ldots, 19, 20$, there are 6 multiples of 3.
Therefore, in the list $1^2, 2^2, 3^2, \ldots, 19^2, 20^2$, there are 6 multiples of 9.

ANSWER: (E)

13. In an isosceles right-angled triangle, the ratio of the length of the hypotenuse to the length of each of the shorter sides is $\sqrt{2} : 1$.
Consider $\triangle WZX$ which is isosceles and right-angled at Z.
Here, $WX : WZ = \sqrt{2} : 1$. Since $WX = 6\sqrt{2}$, then $WZ = \dfrac{6\sqrt{2}}{\sqrt{2}} = 6$.
Since $\triangle WZX$ is isosceles, then $XZ = WZ = 6$.
Consider $\triangle XYZ$ which is isosceles and right-angled at Y.
Here, $YZ = \dfrac{XZ}{\sqrt{2}} = \dfrac{6}{\sqrt{2}} = \dfrac{6}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$.
Since $\triangle XYZ$ is isosceles, then $XY = YZ = 3\sqrt{2}$.
Therefore, the perimeter of $WXYZ$ is

$$WX + XY + YZ + WZ = 6\sqrt{2} + 3\sqrt{2} + 3\sqrt{2} + 6 = 12\sqrt{2} + 6 \approx 22.97$$

Of the given choices, this is closest to 23.

ANSWER: (C)

14. Suppose that Natascha runs at r km/h.
Since she cycles 3 times as fast as she runs, she cycles at $3r$ km/h.
In 1 hour of running, Natascha runs $(1\text{ h}) \cdot (r\text{ km/h}) = r$ km.
In 4 hours of cycling, Natascha cycles $(4\text{ h}) \cdot (3r\text{ km/h}) = 12r$ km.
Thus, the ratio of the distance that she cycles to the distance that she runs is equivalent to the ratio $12r$ km : r km which is equivalent to $12 : 1$.

Answer: (A)

15. *Solution 1*
Since a is a positive integer, $45a$ is a positive integer.
Since b is a positive integer, $45a$ is less than 2021.
The largest multiple of 45 less than 2021 is $45 \times 44 = 1980$. (Note that $45 \cdot 45 = 2025$ which is greater than 2021.)
If $a = 44$, then $b = 2021 - 45 \cdot 44 = 41$.
Here, $a + b = 44 + 41 = 85$.
If a is decreased by 1, the value of $45a + b$ is decreased by 45 and so b must be increased by 45 to maintain the same value of $45a + b$, which increases the value of $a + b$ by $-1 + 45 = 44$.
Therefore, if $a < 44$, the value of $a + b$ is always greater than 85.
If $a > 44$, then $45a > 2021$ which makes b negative, which is not possible.
Therefore, the minimum possible value of $a + b$ is 85.

Solution 2
We re-write $45a + b = 2021$ as $44a + (a + b) = 2021$.
Since a and b are positive integers, $44a$ and $a + b$ are positive integers.
In particular, this tells us that $44a$, which is a multiple of 44, is less than 2021.
Since the sum of $44a$ and $a + b$ is constant, to minimize $a + b$, we can try to maximize $44a$.
Since $44 \cdot 45 = 1980$ and $44 \cdot 46 = 2024$, the largest multiple of 44 less than 2021 is 1980.
This means that $a + b \geq 2021 - 1980 = 41$.
However, $a+b$ cannot equal 41 since we would need $44a = 1980$ and so $a = 45$ (making $b = -4$) to make this possible.
The next multiple of 44 less than 1980 is $44 \cdot 44 = 1936$.
If $a = 44$, then $a + b = 2021 - 44a = 85$.
If $a = 44$ and $a + b = 85$, then $b = 41$ which is possible.
Since $a + b = 41$ is not possible and 85 is the next smallest possible value for $a + b$, then the minimum possible value for $a + b$ is 85.

Answer: (C)

16. The first few values of $n!$ are

$$\begin{aligned}
1! &= 1 \\
2! &= 2(1) = 2 \\
3! &= 3(2)(1) = 6 \\
4! &= 4(3)(2)(1) = 24 \\
5! &= 5(4)(3)(2)(1) = 120
\end{aligned}$$

We note that

$$\begin{aligned} 2! - 1! &= 1 \\ 4! - 1! &= 23 \\ 3! - 1! &= 5 \\ 5! - 1! &= 119 \end{aligned}$$

This means that if a and b are positive integers with $b > a$, then 1, 3, 5, 9 are all possible ones (units) digits of $b! - a!$.
This means that the only possible answer is choice (D), or 7.

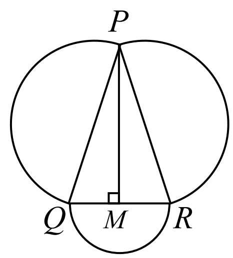
To be complete, we explain why 7 cannot be the ones (units) digit of $b! - a!$.
For $b! - a!$ to be odd, one of $b!$ and $a!$ is even and one of them is odd.
The only odd factorial is 1!, since every other factorial has a factor of 2.
Since $b > a$, then if one of a and b is 1, we must have $a = 1$.
For the ones (units) digit of $b! - 1$ to be 7, the ones (units) digit of $b!$ must be 8.
This is impossible as the first few factorials are shown above and every greater factorial has a ones (units) digit of 0, because it is a multiple of both 2 and 5.

Answer: (D)

17. Since the average of the two smallest integers in S is 5, their sum is $2 \cdot 5 = 10$.
Since the average of the two largest integers in S is 22, their sum is $2 \cdot 22 = 44$.
Suppose that the other five integers in the set S are $p < q < r < t < u$. (Note that the integers in S are all distinct.)
The average of the nine integers in S is thus equal to $\dfrac{10 + 44 + p + q + r + t + u}{9}$ which equals $6 + \dfrac{p + q + r + t + u}{9}$.
We would like this average to be as large as possible.
To make this average as large as possible, we want $\dfrac{p + q + r + t + u}{9}$ to be as large as possible, which means that we want $p + q + r + t + u$ to be as large as possible.
What is the maximum possible value of u?
Let x and y be the two largest integers in S, with $x < y$. Since x and y are the two largest integers, then $u < x < y$.
Since $x + y = 44$ and $x < y$ and x and y are integers, then $x \leq 21$.
For u to be as large as possible (which will allow us to make p, q, r, t as large as possible), we set $x = 21$.
In this case, we can have $u = 20$.
To make p, q, r, t as large as possible, we can take $p = 16$, $q = 17$, $r = 18$, $t = 19$.
Here, $p + q + r + t + u = 90$.
If $x < 21$, then $p + q + r + t + u$ will be smaller and so not give the maximum possible value.
This means that the maximum possible average of the integers in S is $6 + \dfrac{90}{9} = 16$.

Answer: (B)

18. Suppose that $PQ = PR = 2x$ and $QR = 2y$.
The semi-circles with diameters PQ and PR thus have radii x and the radius of the semi-circle with diameter QR is y.
The area of each semi-circle with radius x is $\frac{1}{2}\pi x^2$ and the area of the semi-circle with radius y is $\frac{1}{2}\pi y^2$.
Since the sum of the areas of the three semi-circles equals 5 times the area of the semi-circle with diameter QR, then
$$\tfrac{1}{2}\pi x^2 + \tfrac{1}{2}\pi x^2 + \tfrac{1}{2}\pi y^2 = 5 \cdot \tfrac{1}{2}\pi y^2$$
which gives $x^2 + x^2 + y^2 = 5y^2$ and so $2x^2 = 4y^2$ which gives $x^2 = 2y^2$ and so $x = \sqrt{2}y$.
Suppose that M is the midpoint of QR and that P is joined to M.



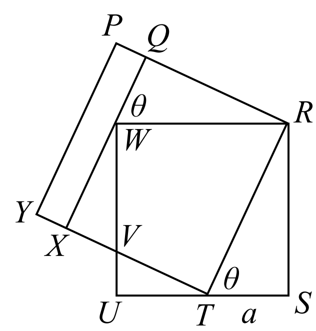
Since $\triangle PQR$ is isosceles with $PQ = PR$, then PM is perpendicular to QR.
In other words, $\triangle PMQ$ is right-angled at M.
Therefore, $\cos(\angle PQR) = \cos(\angle PQM) = \dfrac{QM}{PQ} = \dfrac{\frac{1}{2}QR}{PQ} = \dfrac{y}{2x} = \dfrac{y}{2\sqrt{2}y} = \dfrac{1}{2\sqrt{2}} = \dfrac{1}{\sqrt{4\cdot 2}} = \dfrac{1}{\sqrt{8}}$.

Answer: (B)

19. Since $x + y = 7$, then $x + y + z = 7 + z$.
Thus, the equation $(x + y + z)^2 = 4$ becomes $(7 + z)^2 = 4$.
Since the square of $7 + z$ equals 4, then $7 + z = 2$ or $7 + z = -2$.
If $7 + z = 2$, then $z = -5$.
In this case, since $xz = -180$, we get $x = \dfrac{-180}{-5} = 36$ which gives $y = 7 - x = -29$.
If $7 + z = -2$, then $z = -9$.
In this case, since $xz = -180$, we get $x = \dfrac{-180}{-9} = 20$ which gives $y = 7 - x = -13$.
We can check by direct substitution that $(x, y, z) = (36, -29, -5)$ and $(x, y, z) = (20, -13, -9)$ are both solutions to the original system of equations.
Since S is the sum of the possible values of y, we get $S = (-29)+(-13) = -42$ and so $-S = 42$.

Answer: (E)

20. Let $ST = a$ and let $\angle STR = \theta$.
Since $\triangle RST$ is right-angled at S, then $\angle TRS = 90^\circ - \theta$.
Since $PRTY$ and $WRSU$ are squares, then $\angle PRT = \angle WRS = 90^\circ$.
Thus, $\angle QRW + \angle WRT = \angle WRT + \angle TRS$ and so $\angle QRW = \angle TRS = 90^\circ - \theta$.
Since $PQXY$ is a rectangle, then $\angle PQX = 90^\circ$, which means that $\triangle WQR$ is right-angled at Q.
This means that $\angle QWR = 90^\circ - \angle QRW = 90^\circ - (90^\circ - \theta) = \theta$.



Consider $\triangle RST$.
Since $ST = a$ and $\angle STR = \theta$, then $\cos\theta = \dfrac{a}{RT}$ and so $RT = \dfrac{a}{\cos\theta}$.
Also, $\tan\theta = \dfrac{RS}{a}$ and so $RS = a\tan\theta$.
Since $PRTY$ is a square, then $PY = PR = RT = \dfrac{a}{\cos\theta}$.
Since $WRSU$ is a square, then $RW = RS = a\tan\theta$.
Next, consider $\triangle QRW$.
Since $RW = a\tan\theta$ and $\angle QWR = \theta$, then $\sin\theta = \dfrac{QR}{a\tan\theta}$ and so $QR = a\tan\theta\sin\theta$.
This means that

$$\begin{aligned}
PQ &= PR - QR \\
&= \frac{a}{\cos\theta} - a\tan\theta\sin\theta \\
&= \frac{a}{\cos\theta} - a\cdot\frac{\sin\theta}{\cos\theta}\cdot\sin\theta \\
&= \frac{a}{\cos\theta} - \frac{a\sin^2\theta}{\cos\theta} \\
&= \frac{a(1-\sin^2\theta)}{\cos\theta} \\
&= \frac{a\cos^2\theta}{\cos\theta} \qquad (\text{since } \sin^2\theta + \cos^2\theta = 1) \\
&= a\cos\theta \qquad (\text{since } \cos\theta \neq 0)
\end{aligned}$$

Since the area of rectangle $PQXY$ is 30, then $PQ \cdot PY = 30$ and so $a\cos\theta \cdot \dfrac{a}{\cos\theta} = 30$ which gives $a^2 = 30$.
Since $a > 0$, we get $a = \sqrt{30} \approx 5.48$. Of the given choices, $ST = a$ is closest to 5.5.

Answer: (C)

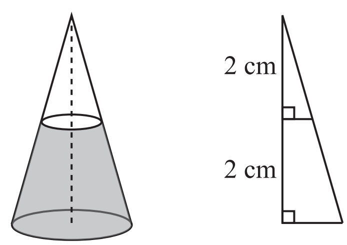
21. Since $f(2) = 5$ and $f(mn) = f(m) + f(n)$, then $f(4) = f(2 \cdot 2) = f(2) + f(2) = 10$.
Since $f(3) = 7$, then $f(12) = f(4 \cdot 3) = f(4) + f(3) = 10 + 7 = 17$.

While this answers the question, is there actually a function that satisfies the requirements? The answer is yes.
One function that satisfies the requirements of the problem is the function f defined by $f(1) = 0$ and $f(2^p 3^q r) = 5p + 7q$ for all non-negative integers p and q and all positive integers r that are not divisible by 2 or by 3. Can you see why this function satisfies the requirements?

Answer: (A)

22. The total surface area of the cone includes the circular base and the lateral surface.
For the given unpainted cone, the base has area $\pi r^2 = \pi(3\text{ cm})^2 = 9\pi\text{ cm}^2$ and the lateral surface has area $\pi rs = \pi(3\text{ cm})(5\text{ cm}) = 15\pi\text{ cm}^2$.
Thus, the total surface area of the unpainted cone is $9\pi\text{ cm}^2 + 15\pi\text{ cm}^2 = 24\pi\text{ cm}^2$.
Since the height and base are perpendicular, the lengths s, h and r form a right-angled triangle with hypotenuse s.
For this cone, by the Pythagorean Theorem, $h^2 = s^2 - r^2 = (5\text{ cm})^2 - (3\text{ cm})^2 = 16\text{ cm}^2$ and so $h = 4$ cm.

When the unpainted cone is placed in the container of paint so that the paint rises to a depth of 2 cm, the base of the cone (area $9\pi\text{ cm}^2$) is covered in paint.
Also, the bottom portion of the lateral surface is covered in paint.



The unpainted portion of the cone is itself a cone with height 2.
When we take a vertical cross-section of the cone through its top vertex and a diameter of the base, the triangle formed above the paint is similar to the original triangle and has half its dimensions. The triangles are similar because both are right-angled and they share an equal angle at the top vertex. The ratio is 2 : 1 since their heights are 4 cm and 2 cm.
Therefore, the unpainted cone has radius 1.5 cm (half of the original radius of 3 cm) and slant height 2.5 cm (half of the original slant height of 5 cm).
Thus, the unpainted lateral surface area is the lateral surface area of a cone with radius 1.5 cm and slant height 2.5 cm, and so has area $\pi(1.5\text{ cm})(2.5\text{ cm}) = 3.75\pi\text{ cm}^2$.
This means that the painted lateral surface area is $15\pi\text{ cm}^2 - 3.75\pi\text{ cm}^2 = 11.25\pi\text{ cm}^2$. (This is in fact three-quarters of the total surface area. Can you explain why this is true in a different way?)
Thus, the fraction of the total surface area of the cone that is painted is

$$\frac{9\pi\text{ cm}^2 + 11.25\pi\text{ cm}^2}{24\pi\text{ cm}^2} = \frac{20.25}{24} = \frac{81}{96} = \frac{27}{32}$$

Since $\frac{27}{32}$ is in lowest terms, then $p + q = 27 + 32 = 59$.

Answer: (A)

23. Since each Figure is formed by placing two copies of the previous Figure side-by-side along the base and then adding other pieces above, the number of dots in the base of each Figure is two times as many as in the previous Figure.
Since each Figure is an equilateral triangle, then the number of dots in the Figure equals the sum of the positive integers from 1 to the number of dots in the base, inclusive. In other words, if the base of a Figure consists of b dots, then the Figure includes $1 + 2 + 3 + \cdots + (b-1) + b$ dots. This sum is equal to $\frac{1}{2}b(b+1)$. (If this formula for the sum is unfamiliar, can you argue why it is true?)
Since each Figure is formed by using three copies of the previous Figure and any new dots added are shaded dots, the number of unshaded dots in each Figure is exactly three times the number of unshaded dots in the previous Figure.
Since each dot is either shaded or unshaded, the number of shaded dots equals the total number of dots minus the number of unshaded dots.
Using these statements, we construct a table:

Figure	Dots in base	Dots in Figure	Unshaded dots	Shaded dots
1	2	3	3	0
2	4	10	9	1
3	8	36	27	9
4	16	136	81	55
5	32	528	243	285
6	64	2080	729	1351
7	128	8256	2187	6069
8	256	32 896	6561	26 335
9	512	131 328	19 683	111 645

Therefore, the smallest value of n for which Figure n includes at least 100 000 dots is $n = 9$.

We note that since the number of dots in the base of Figure 1 is 2 and the number of dots in the base of each subsequent Figure is double the number of dots in the previous Figure, then the number of dots in the base of Figure n is equal to 2^n.
Since the number of unshaded dots in Figure 1 is 3 and the number of unshaded dots in each subsequent Figure is three times the number of unshaded dots in the previous Figure, then the number of unshaded dots in Figure n is 3^n.
Therefore, a formula for the number of unshaded dots in Figure n is $\frac{1}{2}2^n(2^n+1) - 3^n$ which can be re-written as $2^{2n-1} + 2^{n-1} - 3^n$, which agrees with the numbers in the table above.

ANSWER: (B)

24. The curves with equations $y = ax^2 + 2bx - a$ and $y = x^2$ intersect exactly when the equation

$$ax^2 + 2bx - a = x^2$$

has at least one real solution.
This equation has at least one real solution when the quadratic equation

$$(a-1)x^2 + 2bx - a = 0$$

has at least one real solution.
(Note that when $a = 1$, this equation is actually linear as long as $b \neq 0$ and so will have at least one real solution.)

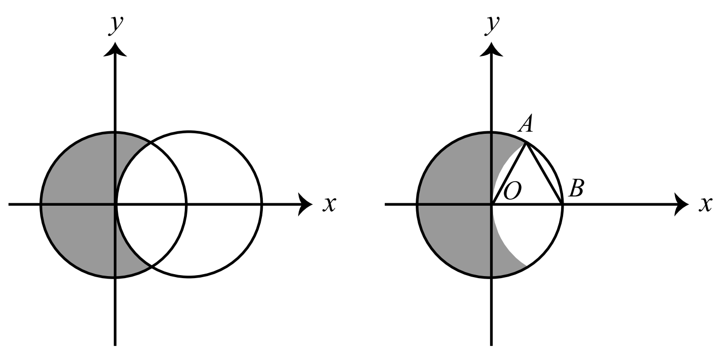
This quadratic equation has at least one real solution when its discriminant is non-negative; that is, when

$$(2b)^2 - 4(a-1)(-a) \geq 0$$

Manipulating algebraically, we obtain the equivalent inequalities:

$$\begin{aligned} 4a^2 - 4a + 4b^2 &\geq 0 \\ a^2 - a + b^2 &\geq 0 \\ a^2 - a + \tfrac{1}{4} + b^2 &\geq \tfrac{1}{4} \\ a^2 - 2a(\tfrac{1}{2}) + (\tfrac{1}{2})^2 + b^2 &\geq \tfrac{1}{4} \\ (a-\tfrac{1}{2})^2 + b^2 &\geq (\tfrac{1}{2})^2 \end{aligned}$$

Therefore, we want to determine the probability that a point (a, b) satisfies $(a-\frac{1}{2})^2+b^2 \geq (\frac{1}{2})^2$ given that it satisfies $a^2 + b^2 \leq (\frac{1}{2})^2$.
The equation $(x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$ represents a circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.
The inequality $(x-\frac{1}{2})^2 + y^2 \geq (\frac{1}{2})^2$ represents the region outside of this circle.
The equation $x^2 + y^2 = (\frac{1}{2})^2$ represents a circle with centre $(0, 0)$ and radius $\frac{1}{2}$.
The inequality $x^2 + y^2 \leq (\frac{1}{2})^2$ represents the region inside this circle.
Re-phrasing the problem in an equivalent geometric way, we want to determine the probability that a point (a, b) is *outside* the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ given that it is *inside* the circle with centre $(0, 0)$ and radius $\frac{1}{2}$.
Note that each of these two circles passes through the centre of the other circle.



Putting this another way, we want to determine the fraction of the area of the circle centred at the origin that is outside the circle centred at $(\frac{1}{2}, 0)$. This region is shaded in the first diagram above.
Let $r = \frac{1}{2}$.
The area of each circle is πr^2.
To determine the shaded area, we subtract the unshaded area from the area of the circle. By symmetry, the unshaded area below the x-axis will equal the unshaded area above the x-axis.
In the second diagram above, the origin is labelled O, the point where the circle centred at O intersects the x-axis is labelled B, and the point of intersect in the first quadrant of the two cicles is labelled A. The unshaded region above the x-axis consists of $\triangle AOB$ plus two curvilinear regions.
Since OA and OB are radii of the circle centred at O, then $OA = OB = r$.
Since AB is a radius of the circle centred at B, then $AB = r$.
This means that $\triangle OAB$ is equilateral, and so it has three 60° angles.

When $\triangle OAB$ is combined with the curvilinear region above and to the right of $\triangle OAB$, we thus obtain a sector of the circle with centre O and central angle 60°.
Since 60° is $\frac{1}{6}$ of 360°, this sector is $\frac{1}{6}$ of the entire circle and so has area $\frac{1}{6}\pi r^2$.
Similarly, when $\triangle OAB$ is combined with the curvilinear region above and to the left of $\triangle OAB$, we obtain a sector of the circle with centre B and central angle 60°; this sector has area $\frac{1}{6}\pi r^2$.
If K is the area of equilateral $\triangle OAB$, the area of each curvilinear region is $\frac{1}{6}\pi r^2 - K$ and so the unshaded area above the x-axis is equal to $2(\frac{1}{6}\pi r^2 - K) + K$ which simplifies to $\frac{1}{3}\pi r^2 - K$.
Thus, the total unshaded area inside the circle of area πr^2 is $\frac{2}{3}\pi r^2 - 2K$, which means that the shaded area inside the circle is $\pi r^2 - (\frac{2}{3}\pi r^2 - 2K)$ which is equal to $\frac{1}{3}\pi r^2 + 2K$.
Finally, we need to use the area of an equilateral triangle with side length r. This area is equal to $\frac{\sqrt{3}}{4}r^2$. (To see this, we could use an altitude to divide the equilateral triangle into two 30°-60°-90° triangles. Using the ratio of side lengths in these special triangles, this altitude has length $\frac{\sqrt{3}}{2}r$ and so the area of the equilateral triangle is $\frac{1}{2}r(\frac{\sqrt{3}}{2})$ or $\frac{\sqrt{3}}{4}r^2$.)
Therefore, the fraction of the circle that is shaded is

$$\frac{\frac{1}{3}\pi r^2 + \frac{\sqrt{3}}{2}r^2}{\pi r^2} = \frac{\frac{1}{3}\pi + \frac{\sqrt{3}}{2}}{\pi} = \frac{2\pi + 3\sqrt{3}}{6\pi} \approx 0.609$$

This means that the probability, p, that a point (a, b) satisfies the given conditions is approximately 0.609 and so $100p \approx 60.9$.
Of the given choices, this is closest to 61, or (E).

ANSWER: (E)

25. Suppose that a, b and c are positive integers with $abc = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2$.
We determine the number of triples (a, b, c) with this property. (We are temporarily ignoring the size ordering condition in the original question.)
Since the product abc has two factors of 2, then a, b and c have a total of two factors of 2.
There are 6 ways in which this can happen: both factors in a, both factors in b, both in c, one each in a and b, one each in a and c, and one each in b and c.
Similarly, there are 6 ways of distributing each of the other squares of prime factors.
Since abc includes exactly 8 squares of prime factors and each can be distributed in 6 ways, there are 6^8 ways of building triples (a, b, c) using the prime factors, and so there are 6^8 triples (a, b, c) with the required product.

Next, we include the condition no pair of a, b and c should be equal. (We note that a, b and c cannot all be equal, since their product is not a perfect cube.)
We count the number of triples with one pair equal, and subtract this number from 6^8.
We do this by counting the number of these triples with $a = b$. By symmetry, the number of triples with $a = c$ and with $b = c$ will be equal to this total.
In order to have $a = b$ and $a \neq c$ and $b \neq c$, for each of the squared prime factors p^2 of abc, either p^2 is distributed as p and p in each of a and b, or p^2 is distributed to c.
Thus, for each of the 8 squared prime factors p^2, there are 2 ways to distribute, and so 2^8 triples (a, b, c) with $a = b$ and $a \neq c$ and $b \neq c$.
Similarly, there will be 2^8 triples with $a = c$ and 2^8 triples with $b = c$.
This means that there are $6^8 - 3 \cdot 2^8$ triples (a, b, c) with the required product and with no two of a, b, c equal.

The original problem asked us to the find the number of triples (x, y, z) with the given product and with $x < y < z$.
To convert triples (a, b, c) with no size ordering to triples (x, y, z) with $x < y < z$, we divide

by 6. (Each triple (x, y, z) corresponds to 6 triples (a, b, c) of distinct positive integers with no size ordering.)
Therefore, the total number of triples (x, y, z) with the required properties is

$$N = \tfrac{1}{6}(6^8 - 3 \cdot 2^8) = 6^7 - 2^7 = 279\,808$$

When N is divided by 100, the remainder is 8.

Answer: (C)