The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Fermat Contest

(Grade 11)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 60 minutes

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Instructions

1. Do not open the Contest booklet until you are told to do so.
2. You may use rulers, compasses and paper for rough work.
3. Be sure that you understand the coding system for your response form. If you are not sure, ask your teacher to clarify it. All coding must be done with a pencil, preferably HB. Fill in circles completely.
4. On your response form, print your school name and city/town in the box in the upper right corner.
5. **Be certain that you code your name, age, grade, and the Contest you are writing in the response form. Only those who do so can be counted as eligible students.**
6. This is a multiple-choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. After making your choice, fill in the appropriate circle on the response form.
7. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
   There is *no penalty* for an incorrect answer.
   Each unanswered question is worth 2, to a maximum of 10 unanswered questions.
8. Diagrams are *not* drawn to scale. They are intended as aids only.
9. When your supervisor tells you to begin, you will have *sixty* minutes of working time.
10. You may not write more than one of the Pascal, Cayley and Fermat Contests in any given year.

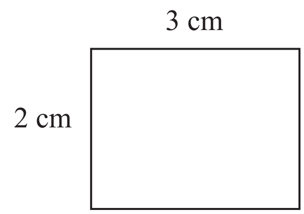
Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

Scoring:	There is *no penalty* for an incorrect answer. Each unanswered question is worth 2, to a maximum of 10 unanswered questions.

Part A: Each correct answer is worth 5.

1. A rectangle has width 2 cm and length 3 cm.
The area of the rectangle is



(A) 2 cm^2 **(B)** 9 cm^2 **(C)** 5 cm^2
(D) 36 cm^2 **(E)** 6 cm^2

2. The expression $2 + 3 \times 5 + 2$ equals

(A) 19 **(B)** 27 **(C)** 35 **(D)** 17 **(E)** 32

3. The number equal to 25% of 60 is

(A) 10 **(B)** 15 **(C)** 20 **(D)** 12 **(E)** 18

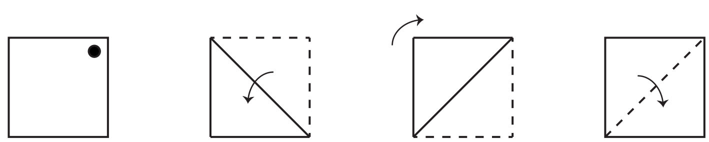
4. When $x = 2021$, the value of $\dfrac{4x}{x+2x}$ is

(A) $\frac{3}{4}$ **(B)** $\frac{4}{3}$ **(C)** 2021 **(D)** 2 **(E)** 6

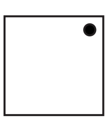
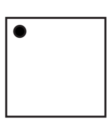
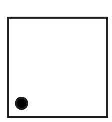
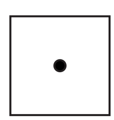
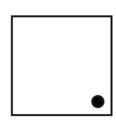
5. Which of the following integers *cannot* be written as a product of two integers, each greater than 1?

(A) 6 **(B)** 27 **(C)** 53 **(D)** 39 **(E)** 77

6. A square piece of paper has a dot in its top right corner and is lying on a table. The square is folded along its diagonal, then rotated 90° clockwise about its centre, and then finally unfolded, as shown.



The resulting figure is

(A)  **(B)**  **(C)**  **(D)**  **(E)** 

7. For which of the following values of x is x greater than x^2 ?

(A) $x = -2$ **(B)** $x = -\frac{1}{2}$ **(C)** $x = 0$ **(D)** $x = \frac{1}{2}$ **(E)** $x = 2$

8. The digits in a two-digit positive integer are reversed. The new two-digit integer minus the original integer equals 54. What is the positive difference between the two digits of the original integer?

(A) 5 **(B)** 7 **(C)** 6 **(D)** 8 **(E)** 9

9. The line with equation $y = 2x - 6$ is translated upwards by 4 units. (That is, every point on the line is translated upwards by 4 units, forming a new line.) The x-intercept of the resulting line is

(A) 3 **(B)** $\frac{3}{2}$ **(C)** 4 **(D)** 1 **(E)** 2

10. If $3^x = 5$, the value of 3^{x+2} is

(A) 10 **(B)** 25 **(C)** 2187 **(D)** 14 **(E)** 45

Part B: Each correct answer is worth 6.

11. In the sum shown, P, Q and R represent three different single digits. The value of $P + Q + R$ is

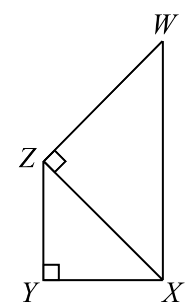
$$\begin{array}{cccc} & P & 7 & R \\ + & 3 & 9 & R \\ \hline & R & Q & 0 \end{array}$$

(A) 13 **(B)** 12 **(C)** 14
(D) 3 **(E)** 4

12. How many of the 20 perfect squares $1^2, 2^2, 3^2, \ldots, 19^2, 20^2$ are divisible by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

13. In the diagram, each of $\triangle WXZ$ and $\triangle XYZ$ is an isosceles right-angled triangle. The length of WX is $6\sqrt{2}$. The perimeter of quadrilateral $WXYZ$ is closest to



(A) 18 **(B)** 20 **(C)** 23
(D) 25 **(E)** 29

14. Natascha cycles 3 times as fast as she runs. She spends 4 hours cycling and 1 hour running. The ratio of the distance that she cycles to the distance that she runs is

(A) $12 : 1$ **(B)** $7 : 1$ **(C)** $4 : 3$ **(D)** $16 : 9$ **(E)** $1 : 1$

15. Let a and b be positive integers for which $45a + b = 2021$. The minimum possible value of $a + b$ is

(A) 44 **(B)** 82 **(C)** 85 **(D)** 86 **(E)** 130

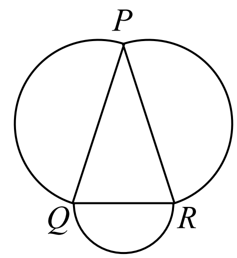
16. If n is a positive integer, the notation $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $4! = 4(3)(2)(1) = 24$ and $1! = 1$. If a and b are positive integers with $b > a$, the ones (units) digit of $b! - a!$ cannot be

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

17. The set S consists of 9 distinct positive integers. The average of the two smallest integers in S is 5. The average of the two largest integers in S is 22. What is the greatest possible average of all of the integers of S?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

18. In the diagram, $\triangle PQR$ is an isosceles triangle with $PQ = PR$. Semi-circles with diameters PQ, QR and PR are drawn. The sum of the areas of these three semi-circles is equal to 5 times the area of the semi-circle with diameter QR. The value of $\cos(\angle PQR)$ is



(A) $\frac{1}{3}$ **(B)** $\frac{1}{\sqrt{8}}$ **(C)** $\frac{1}{\sqrt{12}}$

(D) $\frac{1}{\sqrt{15}}$ **(E)** $\frac{1}{\sqrt{10}}$

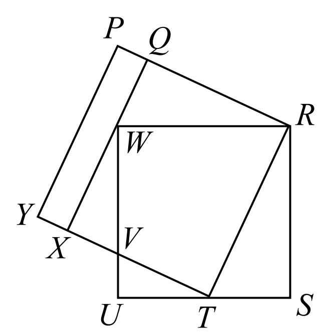
19. The real numbers x, y and z satisfy the three equations

$$x + y = 7$$
$$xz = -180$$
$$(x + y + z)^2 = 4$$

If S is the sum of the two possible values of y, then $-S$ equals

(A) 56 **(B)** 14 **(C)** 36 **(D)** 34 **(E)** 42

20. In the diagram, $PRTY$ and $WRSU$ are squares. Point Q is on PR and point X is on TY so that $PQXY$ is a rectangle. Also, point T is on SU, point W is on QX, and point V is the point of intersection of UW and TY, as shown. If the area of rectangle $PQXY$ is 30, the length of ST is closest to



(A) 5 **(B)** 5.25 **(C)** 5.5

(D) 5.75 **(E)** 6

Part C: Each correct answer is worth 8.

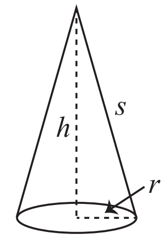
21. A function, f, has $f(2) = 5$ and $f(3) = 7$. In addition, f has the property that

$$f(m) + f(n) = f(mn)$$

for all positive integers m and n. (For example, $f(9) = f(3) + f(3) = 14$.) The value of $f(12)$ is

(A) 17 **(B)** 35 **(C)** 28 **(D)** 12 **(E)** 25

22. An unpainted cone has radius 3 cm and slant height 5 cm. The cone is placed in a container of paint. With the cone's circular base resting flat on the bottom of the container, the depth of the paint in the container is 2 cm. When the cone is removed, its circular base and the lower portion of its lateral surface are covered in paint. The fraction of the total surface area of the cone that is covered in paint can be written as $\frac{p}{q}$ where p and q are positive integers with no common divisor larger than 1. What is the value of $p+q$?



(The *lateral surface* of a cone is its external surface not including the circular base. A cone with radius r, height h, and slant height s has lateral surface area equal to πrs.)

(A) 59 **(B)** 61 **(C)** 63 **(D)** 65 **(E)** 67

23. In Figure 1, three unshaded dots are arranged to form an equilateral triangle, as shown. Figure 2 is formed by arranging three copies of Figure 1 to form the outline of a larger equilateral triangle and then filling the resulting empty space with 1 shaded dot. For each integer $n > 2$, Figure n is formed by first arranging three copies of Figure $n-1$ to form the outline of a larger equilateral triangle and then filling the resulting empty space in the centre with an inverted triangle of shaded dots.

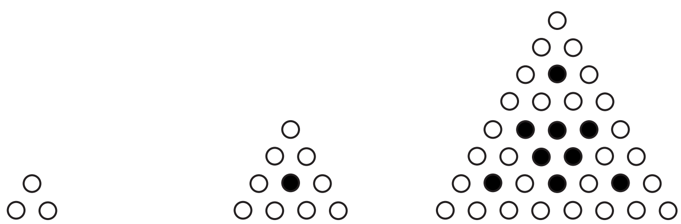


Figure 1 Figure 2 Figure 3

The smallest value of n for which Figure n includes at least 100 000 shaded dots is

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

24. A pair of real numbers (a, b) with $a^2 + b^2 \leq \frac{1}{4}$ is chosen at random. If p is the probability that the curves with equations $y = ax^2 + 2bx - a$ and $y = x^2$ intersect, then $100p$ is closest to

(A) 65 **(B)** 69 **(C)** 53 **(D)** 57 **(E)** 61

25. Let N be the number of triples (x, y, z) of positive integers such that $x < y < z$ and $xyz = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2$. When N is divided by 100, the remainder is

(A) 28 **(B)** 88 **(C)** 8 **(D)** 68 **(E)** 48

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For students...

Thank you for writing the 2021 Fermat Contest! Each year, more than 265 000 students from more than 80 countries register to write the CEMC's Contests.

Encourage your teacher to register you for the Hypatia Contest which will be written in April.

Visit our website `cemc.uwaterloo.ca` to find

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- Information about careers in and applications of mathematics and computer science

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