The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Cayley Contest

(Grade 10)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{2+4}{1+2} = \dfrac{6}{3} = 2$.

Answer: (C)

2. Since $542 \times 3 = 1626$, the ones digit of the result is 6.
Note that the ones digit of a product only depends on the ones digits of the numbers being multiplied, so we could in fact multiply 2×3 and look at the ones digit of this product.

Answer: (E)

3. The top, left and bottom unit squares each contribute 3 sides of length 1 to the perimeter.
The remaining square contributes 1 side of length 1 to the perimeter.
Therefore, the perimeter is $3 \times 3 + 1 \times 1 = 10$.
Alternatively, the perimeter includes 3 vertical right sides, 3 vertical left sides, 2 horizontal top sides, and 2 horizontal bottom sides, which means that the perimeter is $3 + 3 + 2 + 2 = 10$.

Answer: (A)

4. If $3x + 4 = x + 2$, then $3x - x = 2 - 4$ and so $2x = -2$, which gives $x = -1$.

Answer: (D)

5. *Solution 1*
10% of 500 is $\frac{1}{10}$ or 0.1 of 500, which equals 50.
100% of 500 is 500.
Thus, 110% of 500 equals $500 + 50$, which equals 550.

Solution 2
110% of 500 is equal to $\dfrac{110}{100} \times 500 = 110 \times 5 = 550$.

Answer: (E)

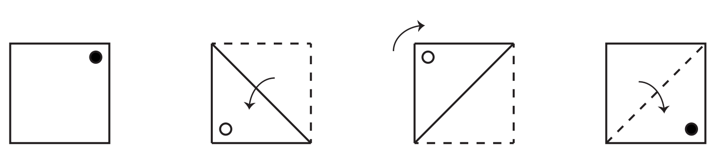
6. Since Eugene swam three times and had an average swim time of 34 minutes, he swam for $3 \times 34 = 102$ minutes in total.
Since he swam for 30 minutes and 45 minutes on Monday and Tuesday, then on Sunday, he swam for $102 - 30 - 45 = 27$ minutes.

Answer: (C)

7. If $x = 1$, then $x^2 = 1$ and $x^3 = 1$ and so $x^3 = x^2$.
If $x > 1$, then x^3 equals x times x^2; since $x > 1$, then x times x^2 is greater than x^2 and so $x^3 > x^2$.
Therefore, if x is positive with $x^3 < x^2$, we must have $0 < x < 1$. We note that if $0 < x < 1$, then both x, x^2 and x^3 are all positive, and $x^3 = x^2 \times x < x^2 \times 1 = x^2$.
Of the given choices, only $x = \frac{3}{4}$ satisfies $0 < x < 1$, and so the answer is (B).
We can verify by direct calculation that this is the only correct answer from the given choices.

Answer: (B)

8. We draw an unshaded dot to represent the location of the dot when it is on the other side of the sheet of paper being shown. Therefore, the dot moves as follows:

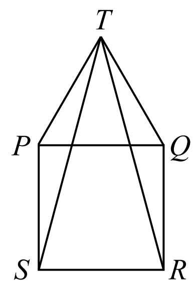


Answer: (E)

9. Suppose that Janice is x years old now.
Two years ago, Janice was $x - 2$ years old.
In 12 years, Janice will be $x + 12$ years old.
From the given information $x + 12 = 8(x - 2)$ and so $x + 12 = 8x - 16$ which gives $7x = 28$ and so $x = 4$.
Checking, if Janice is 4 years old now, then 2 years ago, she was 2 years old and in 12 years she will be 16 years old; since $16 = 2 \times 8$, this is correct.

Answer: (A)

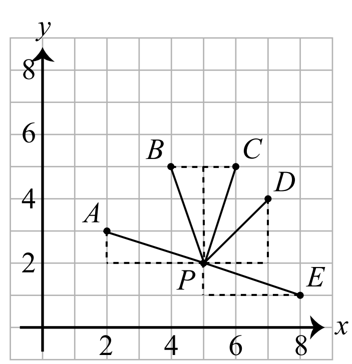
10. Join S to T and R to T.



Since $PQRS$ is a square, $\angle SPQ = 90^\circ$.
Since $\triangle PTQ$ is equilateral, $\angle TPQ = 60^\circ$.
Therefore, $\angle SPT = \angle SPQ + \angle TPQ = 90^\circ + 60^\circ$.
Since $PQRS$ is a square, $SP = PQ$.
Since $\triangle PTQ$ is equilateral, $TP = PQ$.
Since $SP = PQ$ and $TP = PQ$, then $SP = TP$ which means that $\triangle SPT$ is isosceles.
Thus, $\angle PTS = \frac{1}{2}(180^\circ - \angle SPT) = \frac{1}{2}(180^\circ - 150^\circ) = 15^\circ$.
Using a similar argument, we can show that $\angle QTR = 15^\circ$.
This means that $\angle STR = \angle PTQ - \angle PTS - \angle QTR = 60^\circ - 15^\circ - 15^\circ = 30^\circ$.

Answer: (D)

11. *Solution 1*
The points A, B, C, and E can each be reached from point P by moving 3 units in either the x- or y-direction and 1 unit in the other direction.
This means that the distance from P to each of these points is $\sqrt{3^2 + 1^2} = \sqrt{10}$, using the Pythagorean Theorem.



Therefore, the distance from P to D must be the distance that is different.

Solution 2

The coordinates of the 6 points are $A(2,3)$, $B(4,5)$, $C(6,5)$, $D(7,4)$, $E(8,1)$, $P(5,2)$.
Therefore, the distances from P to each of the five other points are

$$\begin{aligned}
PA &= \sqrt{(5-2)^2+(2-3)^2} = \sqrt{3^2+(-1)^2} = \sqrt{10}\\
PB &= \sqrt{(5-4)^2+(2-5)^2} = \sqrt{1^2+(-3)^2} = \sqrt{10}\\
PC &= \sqrt{(5-6)^2+(2-5)^2} = \sqrt{(-1)^2+(-3)^2} = \sqrt{10}\\
PD &= \sqrt{(5-7)^2+(2-4)^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{8}\\
PE &= \sqrt{(5-8)^2+(2-1)^2} = \sqrt{(-3)^2+1^2} = \sqrt{10}
\end{aligned}$$

This tells us that the distance from P to D is the one distance that is different.

ANSWER: (D)

12. Since $x=2$ and $y=x^2-5$, then $y=2^2-5=4-5=-1$.
Since $y=-1$ and $z=y^2-5$, then $z=(-1)^2-5=1-5=-4$.

ANSWER: (E)

13. Since PQR forms a straight angle, then

$$x^\circ + y^\circ + x^\circ + y^\circ + x^\circ = 180^\circ$$

which gives $3x+2y=180$.
Since $x+y=76$ and $2x+2y+x=180$, then $2(76)+x=180$ or $x=180-152=28$.

ANSWER: (A)

14. *Solution 1*

To find the x-intercept of the original line, we set $y=0$ to obtain $0=2x-6$ or $2x=6$ and so $x=3$.
When the line is reflected in the y-axis, its x-intercept is reflected in the y-axis to become $x=-3$.

Solution 2

When a line is reflected in the y-axis, its slope changes signs (that is, is multiplied by -1).
When a line is reflected in the y-axis, its y-intercept (which is on the y-axis) does not change.
Thus, when the line with equation $y=2x-6$ is reflected in the y-axis, the equation of the resulting line is $y=-2x-6$.
To find the x-intercept of this line, we set $y=0$ to obtain $0=-2x-6$ or $2x=-6$ and so $x=-3$.

ANSWER: (D)

15. Amy bought and then sold $15n$ avocados.
Since she bought the avocados in groups of 3, she bought $\dfrac{15n}{3}=5n$ groups of 3 avocados.
Since she paid \$2 for every 3 avocados, she paid $\$2\times 5n=\$10n$.
Since she sold the avocados in groups of 5, she sold $\dfrac{15n}{5}=3n$ groups of 5 avocados.
Since she sold every 5 avocados for \$4, she received $\$4\times 3n=\$12n$.
In terms of n, Amy's profit is $\$12n-\$10n=\$2n$.
Since we know that Amy's profit was \$100, we get $\$2n=\$100$ and so $2n=100$ or $n=50$.

ANSWER: (C)

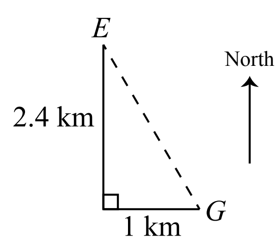
16. Using exponent laws, $3^{x+2} = 3^x \cdot 3^2 = 3^x \cdot 9$.
Since $3^x = 5$, then $3^{x+2} = 3^x \cdot 9 = 5 \cdot 9 = 45$.

Answer: (E)

17. We work backwards through the given information.
At the end, there is 1 candy remaining.
Since $\frac{5}{6}$ of the candies are removed on the fifth day, this 1 candy represents $\frac{1}{6}$ of the candies left at the end of the fourth day.
Thus, there were $6 \times 1 = 6$ candies left at the end of the fourth day.
Since $\frac{4}{5}$ of the candies are removed on the fourth day, these 6 candies represent $\frac{1}{5}$ of the candies left at the end of the third day.
Thus, there were $5 \times 6 = 30$ candies left at the end of the third day.
Since $\frac{3}{4}$ of the candies are removed on the third day, these 30 candies represent $\frac{1}{4}$ of the candies left at the end of the second day.
Thus, there were $4 \times 30 = 120$ candies left at the end of the second day.
Since $\frac{2}{3}$ of the candies are removed on the second day, these 120 candies represent $\frac{1}{3}$ of the candies left at the end of the first day.
Thus, there were $3 \times 120 = 360$ candies left at the end of the first day.
Since $\frac{1}{2}$ of the candies are removed on the first day, these 360 candies represent $\frac{1}{2}$ of the candies initially in the bag.
Thus, there were $2 \times 360 = 720$ in the bag at the beginning.

Answer: (B)

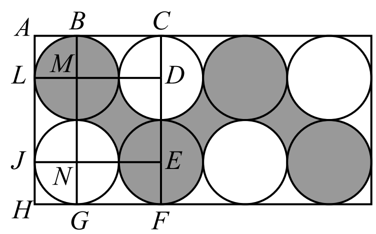
18. Elina and Gustavo start by running and walking for 12 minutes.
Since there are 60 minutes in 1 hour, 12 minutes equals $\frac{1}{5}$ of an hour.
When Elina runs at 12 km/h for $\frac{1}{5}$ of an hour, she runs 12 km/h $\times \frac{1}{5}$ h $= 2.4$ km to the north.
When Gustavo walks at 5 km/h for $\frac{1}{5}$ of an hour, he walks 5 km/h $\times \frac{1}{5}$ h $= 1$ km to the east.
At this point, Elina and Gustavo start to travel directly towards each other.



As they change direction, we use the Pythagorean Theorem to calculate their distance from each other. Using this, we obtain $\sqrt{(2.4\text{ km})^2 + (1\text{ km})^2} = 2.6$ km.
Since Elina continues to travel at 12 km/h, Gustavo continues to travel at 5 km/h, and they travel directly towards each other, they close the gap at a rate of 12 km/h+5 km/h = 17 km/h.
Thus, it takes $\dfrac{2.6\text{ km}}{17\text{ km/h}} \approx 0.153$ h for them to meet.
Since there are 60 minutes in an hour, 0.153 h is equivalent to roughly 9.18 minutes.
Since Elina and Gustavo leave at 3:00 p.m. and travel for 12 minutes and then for an additional 9 minutes, they meet again at approximately 3:21 p.m.

Answer: (E)

19. Each of the four shaded circles has radius 1, and so has area $\pi \cdot 1^2$ which equals π.
Next, we consider one of the three spaces. By symmetry, each of the three spaces has the same area.
Consider the leftmost of the three spaces.
Join the centres of the circles that bound this space to form a quadrilateral. Also, join the centres of these circles to the points where these circles touch the sides of the rectangle.



This quadrilateral is a square with side length 2.
The side length of this quadrilateral is 2, because each of MD, DE, EN, and NM is equal to the sum of lengths of two radii, or 2.
Next, we show that the angles of $DMNE$ are each 90°.
Consider quadrilateral $ABML$. The angle at A is 90°, since the larger shape is a rectangle. The angles at B and L are both 90° since radii are perpendicular to tangents at their points of tangency. Thus, $ABML$ has three 90° angles, which means that its fourth angle must also be 90°.
Now consider quadrilateral $BCDM$. The angles at B and C are both 90°, as above. Since $BM = CD = 1$, this means that $BCMD$ is actually a rectangle.
In a similar way, we can see that $JNGH$ is a square and $MLJN$ is a rectangle.
Finally, we can consider $DMNE$. At M, the three angles outside $DMNE$ are each 90°, which means that the angle at M inside $DMNE$ is 90°.
Similarly, the angle at N inside $DMNE$ is 90°.
Since these angles are both 90° and $MD = NE = 2$, then $DMNE$ is a rectangle. Since its four sides each have length 2, then this rectangle must be a square.

The area of the space between the four circles is thus equal to the area of the square minus the area of the four circular sectors inside the square. In fact, each of these four circular sectors is one-quarter of a circle of radius 1, since its angle at the centre of the circle is 90°.
Thus, the area of the space is equal to $2^2 - 4 \cdot \frac{1}{4} \cdot \pi \cdot 1^2$ which equals $4 - \pi$.
This means that the total area of the shaded region equals $4\pi + 3(4 - \pi) = 4\pi + 12 - 3\pi = 12 + \pi$ which is approximately equal to 15.14.
Of the given choices, this is closest to 15.

Answer: (D)

20. An integer is divisible by both 12 and 20 exactly when it is divisible by the least common multiple of 12 and 20.
The first few positive multiples of 20 are 20, 40, 60. Since 60 is divisible by 12 and neither 20 nor 40 is divisible by 12, then 60 is the least common multiple of 12 and 20.
Since $60 \cdot 16 = 960$ and $60 \cdot 17 = 1020$, the smallest four-digit multiple of 60 is $60 \cdot 17$.
Since $60 \cdot 166 = 9960$ and $60 \cdot 167 = 10\,020$, the largest four-digit multiple of 60 is $60 \cdot 166$.
This means that there are $166 - 17 + 1 = 150$ four-digit multiples of 60.
Now, we need to remove the multiples of 60 that are also multiples of 16.
Since the least common multiple of 60 and 16 is 240, we need to remove the four-digit multiples of 240.
Since $240 \cdot 4 = 960$ and $240 \cdot 5 = 1200$, the smallest four-digit multiple of 240 is $240 \cdot 5$.
Since $240 \cdot 41 = 9840$ and $240 \cdot 42 = 10\,080$, the largest four-digit multiple of 240 is $240 \cdot 41$.
This means that there are $41 - 5 + 1 = 37$ four-digit multiples of 240.
Finally, this means that the number of four-digit integers that are multiples of 12 and 20 but are not multiples of 16 is $150 - 37 = 113$.

Answer: (B)

21. We systematically work through pairs of the given integers to see which pairs add up to a third given integer. Starting with the smallest possible pairs, we have:

$$4 + 27 = 31 \qquad 12 + 15 = 27 \qquad 12 + 27 = 39$$

There are no other pairs that add up to a third given integer.
This means that each of the three sums in the problem must be one of the three sums above.
In the sums above, the only integer that appears three times is 27.
In the sums in the problem, the only variable that appears three times is c.
Therefore, $c = 27$.
This also means that the sum $a + b = c$ must be the sum $12 + 15 = 27$.
Since 12 appears in two sums and 15 does not, then $a = 15$ and $b = 12$.
Matching the values that we know already with the equations that we have, we obtain

$$\begin{array}{ll} a + b = c & 15 + 12 = 27 \\ b + c = d & 12 + 27 = 39 \\ c + e = f & 27 + 4 = 31 \end{array}$$

Therefore, $a + c + f = 15 + 27 + 31 = 73$.

Answer: (C)

22. Suppose that the integer in the bottom left corner is n.
In this case, the sum of the integers in the first column is $64 + 70 + n$ or $n + 134$.
Thus, the sum of the integers in each row, in each column, and on each diagonal also equals $n + 134$. (This means that this square is in fact a magic square, since the sum of the numbers in each row, in each column, and on each diagonal is the same.)
Using the top row, the top right integer equals $(n + 134) - 64 - 10$ or $n + 60$.
Using the northeast diagonal, the centre integer equals $(n + 134) - n - (n + 60)$ or $74 - n$.
Using the second row, the middle integer in the right column equals $(n + 134) - 70 - (74 - n)$ or $2n - 10$.

Using the southeast diagonal, the bottom right integer equals $(n+134) - 64 - (74-n)$ or $2n-4$.
Using the third row, the middle integer is $x = (n+134) - n - (2n-4) = 138 - 2n$.

64	10	$n+60$
70	$74-n$	$2n-10$
n	$138-2n$	$2n-4$

Using the third column,

$$\begin{aligned}(n+60)+(2n-10)+(2n-4) &= n+134\\ 5n+46 &= n+134\\ 4n &= 88\\ n &= 22\end{aligned}$$

Therefore, $x = 130 - 2n = 130 - 44 = 94$ and the complete grid is

64	10	82
70	52	34
22	94	40

.

Answer: (E)

23. Robbie has a score of 8 and Francine has a score of 10 after two rolls each. Thus, in order for Robbie to win (that is, to have a higher total score), his third roll must be at least 3 larger than that of Francine.
If Robbie rolls 1, 2 or 3, his roll cannot be 3 larger than that of Francine.
If Robbie rolls a 4 and wins, then Francine rolls a 1.
If Robbie rolls a 5 and wins, then Francine rolls a 1 or a 2.
If Robbie rolls a 6 and wins, then Francine rolls a 1 or a 2 or a 3.
We now know the possible combinations of rolls that lead to Robbie winning, and so need to calculate the probabilities.
We recall that Robbie and Francine are rolling a special six-sided dice.
Suppose that the probability of rolling a 1 is p.
From the given information, the probability of rolling a 2 is $2p$, of rolling a 3 is $3p$, and of rolling a 4, 5 and 6 is $4p$, $5p$ and $6p$, respectively.
Since the combined probability of rolling a 1, 2, 3, 4, 5, or 6 equals 1, we get the equation $p + 2p + 3p + 4p + 5p + 6p = 1$ which gives $21p = 1$ or $p = \frac{1}{21}$.
Thus, the probability that Robbie rolls a 4 and Francine rolls a 1 equals the product of the probabilities of each of these events, which equals $\frac{4}{21} \cdot \frac{1}{21}$.
Also, the probability that Robbie rolls a 5 and Francine rolls a 1 or 2 equals $\frac{5}{21} \cdot \frac{1}{21} + \frac{5}{21} \cdot \frac{2}{21}$.
Lastly, the probability that Robbie rolls a 6 and Francine rolls a 1, a 2, or a 3 equals

$$\frac{6}{21} \cdot \frac{1}{21} + \frac{6}{21} \cdot \frac{2}{21} + \frac{6}{21} \cdot \frac{3}{21}$$

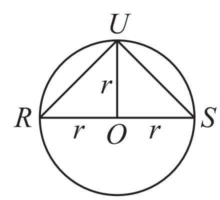
Therefore, the probability that Robbie wins is

$$\frac{4}{21} \cdot \frac{1}{21} + \frac{5}{21} \cdot \frac{1}{21} + \frac{5}{21} \cdot \frac{2}{21} + \frac{6}{21} \cdot \frac{1}{21} + \frac{6}{21} \cdot \frac{2}{21} + \frac{6}{21} \cdot \frac{3}{21} = \frac{4+5+10+6+12+18}{21 \cdot 21} = \frac{55}{441}$$

which is in lowest terms since $55 = 5 \cdot 11$ and $441 = 3^2 \cdot 7^2$.
Converting to the desired form, we see that $r = 55$ and $s = 41$ which gives $r + s = 96$.

Answer: (A)

24. Let O be the centre of the top face of the cylinder and let r be the radius of the cylinder.
We need to determine the value of QT^2.
Since RS is directly above PQ, then RP is perpendicular to PQ.
This means that $\triangle TPQ$ is right-angled at P.
Since PQ is a diameter, then $PQ = 2r$.
By the Pythagorean Theorem, $QT^2 = PT^2 + PQ^2 = n^2 + (2r)^2 = n^2 + 4r^2$.
So we need to determine the values of n and r. We will use the information about QU and UT to determine these values.
Join U to O.
Since U is halfway between R and S, then the arcs RU and US are each one-quarter of the circle that bounds the top face of the cylinder.
This means that $\angle UOR = \angle UOS = 90°$.



We can use the Pythagorean Theorem in $\triangle UOR$ and $\triangle UOS$, which are both right-angled at O, to obtain

$$UR^2 = UO^2 + OR^2 = r^2 + r^2 = 2r^2 \qquad \text{and} \qquad US^2 = 2r^2$$

Since RP and QS are both perpendicular to the top face of the cylinder, we can use the Pythagorean Theorem in $\triangle TRU$ and in $\triangle QSU$ to obtain

$$QU^2 = QS^2 + US^2 = m^2 + 2r^2$$
$$UT^2 = TR^2 + UR^2 = (PR - PT)^2 + 2r^2 = (QS - n)^2 + 2r^2 = (m - n)^2 + 2r^2$$

Since $QU = 9\sqrt{33}$, then $QU^2 = 9^2 \cdot 33 = 2673$.
Since $UT = 40$, then $UT^2 = 1600$.
Therefore,

$$\begin{aligned} m^2 + 2r^2 &= 2673 \\ (m-n)^2 + 2r^2 &= 1600 \end{aligned}$$

Subtracting the second equation from the first, we obtain the equivalent equations

$$\begin{aligned} m^2 - (m-n)^2 &= 1073 \\ m^2 - (m^2 - 2mn + n^2) &= 1073 \\ 2mn - n^2 &= 29 \cdot 37 \\ n(2m - n) &= 29 \cdot 37 \end{aligned}$$

Since m and n are integers, then $2m - n$ is an integer. Thus, n and $2m - n$ are a factor pair of $29 \cdot 37 = 1073$.
Since 29 and 37 are prime numbers, the integer 1073 has only four positive divisors: 1, 29, 37, 1073.
This gives the following possibilities:

n	$2m - n$	$m = \frac{1}{2}((2m - n) + n)$
1	1073	537
29	37	33
37	29	33
1073	1	537

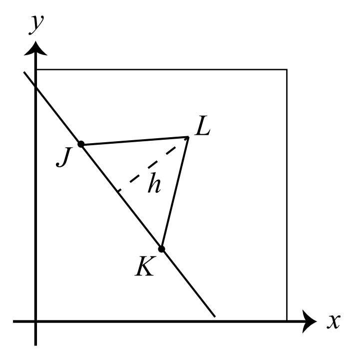
Since $m > n$, then n cannot be 37 or 1073.
Since $QU > QS$, then $m < 9\sqrt{33} \approx 51.7$.
This means that $n = 29$ and $m = 33$.
Since $(m-n)^2 + 2r^2 = 1600$, we obtain $2r^2 = 1600 - (m-n)^2 = 1600 - 4^2 = 1584$ and so

$$QT^2 = n^2 + 4r^2 = 29^2 + 2(2r^2) = 841 + 3168 = 4009$$

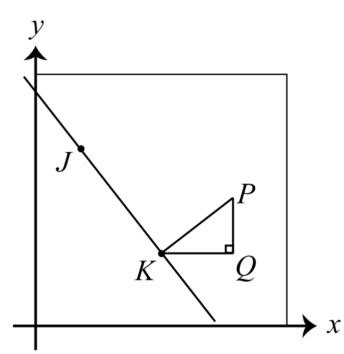
The remainder when QT^2 is divided by 100 is 9.

ANSWER: (C)

25. The distance between $J(2,7)$ and $K(5,3)$ is equal to $\sqrt{(2-5)^2 + (7-3)^2} = \sqrt{3^2 + 4^2} = 5$.
Therefore, if we consider $\triangle JKL$ as having base JK and height h, then we want $\frac{1}{2} \cdot JK \cdot h \leq 10$ which means that $h \leq 10 \cdot \frac{2}{5} = 4$.
In other words, $L(r,t)$ can be any point with $0 \leq r \leq 10$ and $0 \leq t \leq 10$ whose perpendicular distance to the line through J and K is at most 4.



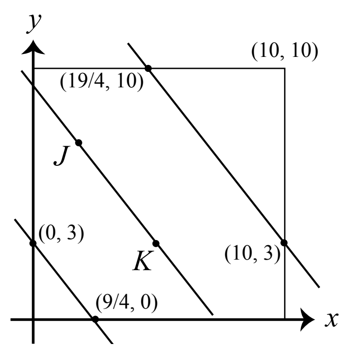
The slope of the line through $J(2,7)$ and $K(5,3)$ is equal to $\frac{7-3}{2-5} = -\frac{4}{3}$.
Therefore, this line has equation $y - 7 = -\frac{4}{3}(x-2)$.
Muliplying through by 3, we obtain $3y - 21 = -4x + 8$ or $4x + 3y = 29$.
We determine the equation of the line above this line that is parallel to it and a perpendicular distance of 4 from it.
The equation of this line will be of the form $4x + 3y = c$ for some real number c, since it is parallel to the line with equation $4x + 3y = 29$.
To determine the value of c, we determine the coordinates of one point on this line.
To determine such a point, we draw a perpendicular of length 4 from K to a point P above the line.
Since JK has slope $-\frac{4}{3}$ and KP is perpendicular to JK, then KP has slope $\frac{3}{4}$.
Draw a vertical line from P and a horizontal line from K, meeting at Q.



Since KP has slope $\frac{3}{4}$, then $PQ : QK = 3 : 4$, which means that $\triangle KQP$ is similar to a 3-4-5 triangle.
Since $KP = 4$, then $PQ = \frac{3}{5}KP = \frac{12}{5}$ and $QK = \frac{4}{3}KP = \frac{16}{5}$.
Thus, the coordinates of P are $\left(5 + \frac{16}{5}, 3 + \frac{12}{5}\right)$ or $\left(\frac{41}{5}, \frac{27}{5}\right)$.
Since P lies on the line with equation $4x + 3y = c$, then

$$c = 4 \cdot \frac{41}{5} + 3 \cdot \frac{27}{5} = \frac{164}{5} + \frac{81}{5} = \frac{245}{5} = 49$$

and so the equation of the line parallel to JK and 4 units above it is $4x + 3y = 49$.
In a similar way, we find that the line parallel to JK and 4 units below it has equation $4x + 3y = 9$. (Note that $49 - 29 = 29 - 9$.)
This gives us the following diagram:



The points L that satisfy the given conditions are exactly the points within the square, below the line $4x+3y = 49$ and above the line $4x+3y = 9$. In other words, the region $\mathcal{R}$ is the region inside the square and between these lines.
To find the area of $\mathcal{R}$, we take the area of the square bounded by the lines $x = 0$, $x = 10$, $y = 0$, and $y = 10$ (this area equals $10 \cdot 10$ or 100) and subtract the area of the two triangles inside the square and not between the lines.
The line with equation $4x + 3y = 9$ intersects the y-axis at $(0, 3)$ (we see this by setting $x = 0$) and the x-axis at $\left(\frac{9}{4}, 0\right)$ (we see this by setting $y = 0$).
The line with equation $4x + 3y = 49$ intersects the line $x = 10$ at $(10, 3)$ (we see this by setting $x = 10$) and the line $y = 10$ at $\left(\frac{19}{4}, 10\right)$ (we see this by setting $y = 10$).
The bottom triangle that is inside the square and outside $\mathcal{R}$ has area $\frac{1}{2} \cdot 3 \cdot \frac{9}{4} = \frac{27}{8}$.
The top triangle that is inside the square and outside $\mathcal{R}$ has horizontal base of length $10 - \frac{19}{4}$ or $\frac{21}{4}$ and vertical height of length $10 - 3$ or 7, and thus has area $\frac{1}{2} \cdot \frac{21}{4} \cdot 7 = \frac{147}{8}$.
Finally, this means that the area of $\mathcal{R}$ is

$$100 - \frac{27}{8} - \frac{147}{8} = 100 - \frac{174}{8} = 100 - \frac{87}{4} = \frac{313}{4}$$

which is in lowest terms since the only divisors of the denominator that are larger than 1 are 2 and 4, while the numerator is odd.

When we write this area in the form $\dfrac{300+a}{40-b}$ where a and b are positive integers, we obtain $a = 13$ and $b = 36$, giving $a + b = 49$.

Answer: (D)