The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Pascal Contest

(Grade 9)

Tuesday, February 25, 2020
(in North America and South America)

Wednesday, February 26, 2020
(outside of North America and South America)

Solutions

1. The figure includes 5 groups of 5 squares.
   Thus, there are $5 \times 5 = 25$ squares in total.

   Answer: (E)

2. Calculating, $0.8 + 0.02 = 0.80 + 0.02 = 0.82$.

   Answer: (C)

3. *Solution 1*
   Since $2x + 6 = 16$, then $\dfrac{2x+6}{2} = \dfrac{16}{2}$ and so $x + 3 = 8$.
   Since $x + 3 = 8$, then $x + 4 = (x + 3) + 1 = 8 + 1 = 9$.

   Solution 2
   Since $2x + 6 = 16$, then $2x = 16 - 6 = 10$.
   Since $2x = 10$, then $\dfrac{2x}{2} = \dfrac{10}{2}$ and so $x = 5$.
   Since $x = 5$, then $x + 4 = 5 + 4 = 9$.

   Answer: (C)

4. The positive divisor pairs of 24 are:

   1 and 24   2 and 12   3 and 8   4 and 6

   Of these, the pair whose sum is 11 is 3 and 8.
   The difference between these two integers is $8 - 3 = 5$.

   Answer: (D)

5. *Solution 1*
   Since the side lengths of the triangle are $x - 1$, $x + 1$ and 7, its perimeter is $(x - 1) + (x + 1) + 7$ which equals $2x + 7$.
   Since $x = 10$, the perimeter equals $2 \times 10 + 7$ which is equal to 27.

   Solution 2
   Since $x = 10$, the side lengths of the triangle are $x - 1 = 9$ and $x + 1 = 11$ and 7.
   The perimeter of the triangle is thus $9 + 11 + 7$ which equals 27.

   Answer: (D)

6. We note that $2^3 = 2 \times 2 \times 2 = 8$ and $2^4 = 2^3 \times 2 = 16$.
   Therefore, $\dfrac{2^4 - 2}{2^3 - 1} = \dfrac{16 - 2}{8 - 1} = \dfrac{14}{7} = 2$.
   Alternatively, $\dfrac{2^4 - 2}{2^3 - 1} = \dfrac{2(2^3 - 1)}{2^3 - 1} = 2$.

   Answer: (E)

7. Ewan's sequence starts with 3 and each following number is 11 larger than the previous number. Since every number in the sequence is some number of 11s more than 3, this means that each number in the sequence is 3 more than a multiple of 11. Furthermore, every such positive integer is in Ewan's sequence.
   Since $110 = 11 \times 10$ is a multiple of 11, then $113 = 110 + 3$ is 3 more than a multiple of 11, and so is in Ewan's sequence.
   Alternatively, we could write Ewan's sequence out until we get into the correct range:

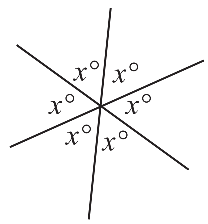
   $$3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, \ldots$$

   Answer: (A)

8. From the bar graph, Matilda saw 6 goldfinches, 9 sparrows, and 5 grackles.
In total, she saw $6 + 9 + 5 = 20$ birds.
This means that the percentage of birds that were goldfinches is $\frac{6}{20} \times 100\% = \frac{3}{10} \times 100\% = 30\%$.

Answer: (C)

9. Since opposite angles are equal, then the three unmarked angles around the central point each has measure x°.



Since the angles around a point add to 360°, then $6 \times x^\circ = 360^\circ$.
From this, $6x = 360$ and so $x = 60$.

Answer: (C)

10. Starting at 1:00 p.m., Jorge watches a movie that is 2 hours and 20 minutes long.
This first movie ends at 3:20 p.m.
Then, Jorge takes a 20 minute break.
This break ends at 3:40 p.m.
Then, Jorge watches a movie that is 1 hour and 45 minutes long.
After 20 minutes of this movie, it is 4:00 p.m. and there is still 1 hour and 25 minutes left in the movie. This second movie thus ends at 5:25 p.m.
Then, Jorge takes a 20 minute break which ends at 5:45 p.m.
Finally, Jorge watches a movie that is 2 hours and 10 minutes long.
This final movie ends at 7:55 p.m.

Answer: (D)

11. 12 and 21 are multiples of 3 ($12 = 4 \times 3$ and $21 = 7 \times 3$) so the answer is not (A) or (D).
16 is a perfect square ($16 = 4 \times 4$) so the answer is not (C).
The sum of the digits of 26 is 8, which is not a prime number, so the answer is not (E).
Since 14 is not a multiple of a three, 14 is not a perfect square, and the sum of the digits of 14 is $1 + 4 = 5$ which is prime, then the answer is 14, which is choice (B).

Answer: (B)

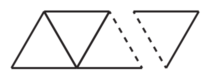
12. Since the average of three heights is 171 cm, then the sum of these three heights is 3×171 cm or 513 cm.
Since Jiayin's height is 161 cm, then the sum of Natalie's and Harpreet's heights must equal 513 cm − 161 cm = 352 cm.
Since Harpreet and Natalie are the same height, this height is $\frac{352 \text{ cm}}{2} = 176$ cm.
Therefore, Natalie's height is 176 cm.

Answer: (C)

13. Since the ratio of apples to bananas is $3:2$, then we can let the numbers of apples and bananas equal $3n$ and $2n$, respectively, for some positive integer n.
Therefore, the total number of apples and bananas is $3n+2n=5n$, which is a multiple of 5.
Of the given choices, only (E) 72 is not a multiple of 5 and so cannot be the total.
(Each of the other choices can be the total by picking an appropriate value of n.)

ANSWER: (E)

14. The first figure consists of one tile with perimeter 3×7 cm $= 21$ cm.
Each time an additional tile is added, the perimeter of the figure increases by 7 cm (one side length of a tile), because one side length of the previous figure is "covered up" and two new side lengths of a tile are added to the perimeter for a net increase of one side length (or 7 cm).



Since the first figure has perimeter 21 cm and we are looking for the figure with perimeter 91 cm, then the perimeter must increase by 91 cm $-$ 21 cm $=$ 70 cm.
Since the perimeter increases by 7 cm when each tile is added, then $\dfrac{70\text{ cm}}{7\text{ cm/tile}} = 10$ tiles need to be added to reach a perimeter of 91 cm.
In total, this figure will have $1+10=11$ tiles.

ANSWER: (B)

15. The total area of the shaded region equals the area of the small square (9) plus the area between the large square and medium square.
Since the area of the large square is 49 and the area of the medium square is 25, then the area of the region between these squares is $49-25=24$.
Therefore, the total area of the shaded region is $9+24=33$.

ANSWER: (A)

16. We look at each of the five choices:

(A) $3(x+2)=3x+6$

(B) $\dfrac{-9x-18}{-3}=\dfrac{-9x}{-3}+\dfrac{-18}{-3}=3x+6$

(C) $\dfrac{1}{3}(3x)+\dfrac{2}{3}(9)=x+6$

(D) $\dfrac{1}{3}(9x+18)=3x+6$

(E) $3x-2(-3)=3x+(-2)(-3)=3x+6$

The expression that is not equivalent to $3x+6$ is the expression from (C).

ANSWER: (C)

17. Since there are two possible prizes that Jamie can win and each is equally likely, then the probability that Jamie wins \$30 is $\frac{1}{2}$ and the probability that Jamie wins \$40 is $\frac{1}{2}$.
If Jamie wins \$30, then for the total value of the prizes to be \$50, Ben must win \$20. The probability that Ben wins \$20 is $\frac{1}{3}$, since there are three equally likely outcomes for Ben.
If Jamie wins \$40, then for the total value of the prizes to be \$50, Ben must win \$10. The probability that Ben wins \$10 is $\frac{1}{3}$.
Since Ben's and Jamie's prizes come from different draws, we can assume that the results are independent, and so the probability that Jamie wins \$30 and Ben wins \$20 is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
Similarly, the probability that Jamie wins \$40 and Ben wins \$10 is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
Therefore, the probability that the total value of their prizes is \$50 is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

ANSWER: (B)

18. Since the square root of n is between 17 and 18, then n is between $17^2 = 289$ and $18^2 = 324$.
Since n is a multiple of 7, we need to count the number of multiples of 7 between 289 and 324.
Since $41 \times 7 = 287$ and $42 \times 7 = 294$, then 294 is the smallest multiple of 7 larger than 289.
Since $46 \times 7 = 322$ and $47 \times 7 = 329$, then 322 is the largest multiple of 7 smaller than 324.
This means that the multiples of 7 between 289 and 324 are $42 \times 7 = 294$, $43 \times 7 = 301$, $44 \times 7 = 308$, $45 \times 7 = 315$, and $46 \times 7 = 322$, of which there are 5.
(We note that we could have determined that there were 5 such multiples by calculating $46 - 42 + 1$ which equals 5.)
Therefore, there are 5 possible values for n.

ANSWER: (D)

19. Each card fits into exactly one of the following categories:

(A) lower case letter on one side, even integer on the other side
(B) lower case letter on one side, odd integer on the other side
(C) upper case letter on one side, even integer on the other side
(D) upper case letter on one side, odd integer on the other side

The given statement is

"If a card has a lower case letter on one side, then it has an odd integer on the other."

If a card fits into category (B), (C) or (D), it does not violate the given statement, and so the given statement is true. If a card fits into category (A), it does violate the given statement. Therefore, we need to turn over any card that might be in category (A). Of the given cards,

(i) 1 card shows a lower case letter and might be in (A),
(ii) 4 cards show an upper case letter and is not in (A),
(iii) 2 cards show an even integer and might be in (A), and
(iv) 8 cards show an odd integer and is not in (A).

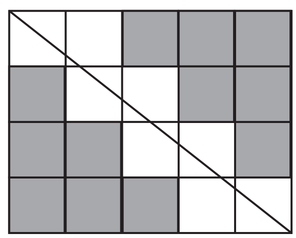
In order to check if this statement is true, we must turn over the cards in (i) and (iii), of which there are 3.

ANSWER: (E)

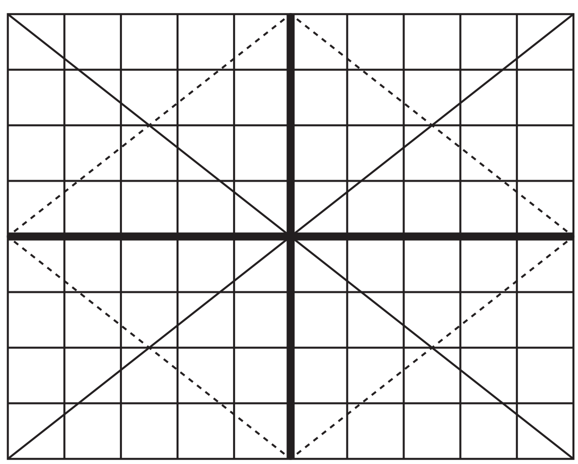
20. The original $5 \times 5 \times 5$ cube has 6 faces, each of which is 5×5.
When the three central columns of cubes is removed, one of the "1×1 squares" on each face is removed.
This means that the surface area of each face is reduced by 1 to $5 \times 5 - 1 = 24$. This means that the total exterior surface area of the cube is $6 \times 24 = 144$.
When each of the central columns is removed, it creates a "tube" that is 5 unit cubes long. Each of these tubes is $5 \times 1 \times 1$.
Since the centre cube of the original $5 \times 5 \times 5$ cube is removed when each of the three central columns is removed, this means that each of the three $5 \times 1 \times 1$ tubes is split into two $2 \times 1 \times 1$ tubes.
The interior surface area of each of these tubes consists of four faces, each of which is 2×1. (We could instead think about the exterior surface area of a $2 \times 1 \times 1$ rectangular prism, ignoring its square ends.) Thus, the interior surface area from 6 tubes each with 4 faces measuring 2×1 gives a total area of $6 \times 4 \times 2 \times 1 = 48$.
In total, the surface area of the resulting solid is $144 + 48 = 192$.

ANSWER: (E)

21. If we remove one diagonal from the given 4×5 grid, we see that 8 squares are intersected by the remaining diagonal and 12 squares are not intersected.



The result is the same whichever of the two diagonals is removed.
We can construct an 8×10 grid with two diagonals by combining four such 4×5 grids:



When these four pieces are joined together, the diagonals of the pieces join to form the diagonal of the large rectangle because their slopes are the same.
In each of the four pieces, 12 of the 1×1 squares are not intersected by either diagonal.
Overall, this means that $4 \times 12 = 48$ of the 1×1 squares are not intersected by either diagonal.

ANSWER: (D)

22. Since point R is on PQ, then $PQ = PR + QR = 6 + 4 = 10$.
Semi-circles with diameters of 10, 6 and 4 have radii of 5, 3 and 2, respectively.
A semi-circle with diameter PQ has area $\frac{1}{2} \times \pi \times 5^2 = \frac{25}{2}\pi$.
A semi-circle with diameter PR has area $\frac{1}{2} \times \pi \times 3^2 = \frac{9}{2}\pi$.
A semi-circle with diameter QR has area $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.
The shaded region consists of a section to the left of PQ and a section to the right of PQ.
The area of the section to the left of PQ equals the area of the semi-circle with diameter PQ minus the area of the semi-circle with diameter PR.
This area is thus $\frac{25}{2}\pi - \frac{9}{2}\pi = 8\pi$.
The area of the section to the right of PQ equals the area of the semi-circle with diameter QR.
This area is thus 2π.
Therefore, the area of the entire shaded region is $8\pi + 2\pi = 10\pi$.
Since the large circle has radius 5, its area is $\pi \times 5^2 = 25\pi$.
Since the area of the shaded region is 10π, then the area of the unshaded region is equal to $25\pi - 10\pi = 15\pi$.
The ratio of the area of the shaded region to the area of the unshaded region is $10\pi : 15\pi$ which is equivalent to $2 : 3$.

Answer: (B)

23. Since there are 4 players in the tournament and each player plays each other player once, then each player plays 3 games.
Since each win earns 5 points and each tie earns 2 points, the possible results for an individual player are:

- 3 wins, 0 losses, 0 ties: 15 points
- 2 wins, 0 losses, 1 tie: 12 points
- 2 wins, 1 loss, 0 ties: 10 points
- 1 win, 0 losses, 2 ties: 9 points
- 1 win, 1 loss, 1 tie: 7 points
- 1 win, 2 losses, 0 ties: 5 points
- 0 wins, 0 losses, 3 ties: 6 points
- 0 wins, 1 loss, 2 ties: 4 points
- 0 wins, 2 losses, 1 tie: 2 points
- 0 wins, 3 losses, 0 ties: 0 points

In the third table given, Deb has 2 points which means that Deb had 1 tie. If one player has a tie, then another player must also have a tie. But neither 15 points nor 5 points is a possible total to obtain with a tie. Therefore, the third table is not possible.

Similarly, in the fourth table, Ali with 12 points must have had a tie, but none of the other players' scores allow for have a tie, so the fourth table is not possible.

In the second table, each of Che and Deb must have 2 ties and neither Ali nor Bea can have a tie because of their totals of 10 points each. Since Che and Deb only played each other once, then each of them must have a tie against another player, which is not possible. Therefore, the second table is not possible.

The first table is possible:

Result	Ali	Bea	Che	Deb
Ali wins against Bea	5 points	0 points		
Ali wins against Che	5 points		0 points	
Ali wins against Deb	5 points			0 points
Bea ties Che		2 points	2 points	
Bea wins against Deb		5 points		0 points
Che ties Deb			2 points	2 points
TOTAL	15 points	7 points	4 points	2 points

Therefore, exactly one of the four given final point distributions is possible.

Answer: (B)

24. If Lucas chooses 1 number only, there are 8 possibilities for the sum, namely the 8 numbers themselves: 2, 5, 7, 12, 19, 31, 50, 81.

To count the number of additional sums to be included when Lucas chooses two numbers, we make a table, adding the number on left to the number on top when it is less than the number on top (we don't need to add the numbers in both directions or a number to itself):

+	2	5	7	12	19	31	50	81
2		7	9	14	21	33	52	83
5			12	17	24	36	55	86
7				19	26	38	57	88
12					31	43	62	93
19						50	69	100
31							81	112
50								131
81								

We note two things from this table. First, any time we add two consecutive numbers from the original list who sum is not too large, we obtain another number in the list. We do not include this as a new sum as these are already accounted for as sums of 1 number only. Second, the remaining sums are all distinct and there are 20 additional sums that are less than or equal to 100.

Lastly, we consider sums formed by three numbers from the list.
The fact that the sum of any two consecutive numbers from the list equals the next number in the list becomes very important in this case.
If the three numbers chosen are three consecutive numbers in the list and their sum is not too large, then their sum is actually equal to the sum of two numbers from the list. This is because the largest two of the three numbers can be combined into one yet larger number from the list. For example, $5 + 7 + 12 = 5 + (7 + 12) = 5 + 19$, which is already counted above.
If any two of the three numbers chosen are consecutive in the list, the same thing happens. For example, $12 + 19 + 50 = (12 + 19) + 50 = 31 + 50$ and $2 + 31 + 50 = 2 + (31 + 50) = 2 + 81$.
Therefore, any additional sums that are created must come from three numbers, no two of which are consecutive.
We count these cases individually and sequentially, knowing that we are only interested in the sums less than 100 and remembering that we cannot include consecutive numbers from the list:

- $2 + 7 + 19 = 28$; $2 + 7 + 31 = 40$; $2 + 7 + 50 = 59$; $2 + 7 + 81 = 90$
- $2 + 12 + 31 = 45$; $2 + 12 + 50 = 64$; $2 + 12 + 81 = 95$
- $2 + 19 + 50 = 71$
- $5 + 12 + 31 = 48$; $5 + 12 + 50 = 67$; $5 + 12 + 81 = 98$
- $5 + 19 + 50 = 74$
- $7 + 19 + 50 = 76$

Every other combination of 3 integers from the list either includes 2 consecutive numbers (and so has been counted already) or includes both 81 and one of 31 and 19 (and so is too large).
In this case, there are 13 additional sums.

Putting the three cases together, there are $8 + 20 + 13 = 41$ different sums less than or equal to 100.

Answer: (E)

25. Before beginning our solution, we need several facts about prime numbers and prime factorizations:

F1. Every positive integer greater than 1 can be written as a product of prime numbers in a unique way. (If the positive integer is itself prime, this product consists of only the prime number.) This fact is called the "Fundamental Theorem of Arithmetic". This fact is often seen implicitly in finding a "factor tree" for a given integer. For example, 1500 is equal to $2^2 \times 3^1 \times 5^3$ and there is no other way of writing 1500 as a product of prime numbers. Note that rearranging the same prime factors in a different order does not count as a different factorization.

F2. If n is a positive integer and d is a positive integer that is a divisor of n, then the only possible prime factors of d are the prime factors of n. For example, if d is a positive divisor of $n = 1500$, then the only possible prime factors of d are 2, 3 and 5. This means, for example, that d cannot be divisible by 7 or by 11 or by any other prime number not equal to 2, 3 or 5. d might or might not be divisible by each of 2, 3 or 5.

F3. If n is a positive integer, d is a positive integer that is a divisor of n, and p is a prime factor of both n and d, then p cannot divide d "more times" than it divides n. For example, if d is a positive divisor of $n = 1500 = 2^2 \times 3^1 \times 5^3$ that is divisible by 5, then d can be divisible by 5 or by 5^2 or by 5^3 but cannot be divisible by 5^4 or by 5^5 or by any larger power of 5.

F4. A positive integer m greater than 1 is a perfect square exactly when each prime power in its prime factorization has an even exponent. For example, $n = 1500 = 2^2 \times 3^1 \times 5^3$ is not a perfect square but $m = 22500 = 2^2 \times 3^2 \times 5^4$ is a perfect square. This is true because if m is a perfect square, then $m = r^2$ for some positive integer r and so we can find the prime factorization of m by writing down the prime factorization of r twice. For example, if $r = 150 = 2^1 \times 3^1 \times 5^2$, then $m = 2^1 \times 3^1 \times 5^2 \times 2^1 \times 3^1 \times 5^2 = 2^2 \times 3^2 \times 5^4$. Further, if the prime factorization of m includes a prime power with an odd exponent, then the copies of this prime number cannot be equally distributed into two copies of the square root of m, which means that $\sqrt{m}$ is not an integer.

F5. One method to find the greatest common divisor (gcd) of two positive integers n and t is to write out the prime factorization of n and t and form a new integer d (the gcd) that is the product, for each common prime divisor, of the largest common prime power that divides both n and t. For example, if $n = 1500 = 2^2 \times 3^1 \times 5^3$ and $t = 7000 = 2^3 \times 5^3 \times 7^1$, then the greatest common divisor of n and t equals $2^2 \times 5^3 = 500$. The justification of this method is based on F2 (since d is a divisor of n and t it can only include prime factors common to both lists) and F3 (since d cannot include a prime power that is too large if it is to be a divisor of each of n and t).

We can now begin our solution.

Suppose that $(205\,800, 35k)$ is a happy pair.
We find the prime factorization of 205 800:

$$\begin{aligned}
205\,800 &= 2058 \times 100 \\
&= 2 \times 1029 \times (2 \times 5)^2 \\
&= 2 \times 3 \times 343 \times 2^2 \times 5^2 \\
&= 2 \times 3 \times 7^3 \times 2^2 \times 5^2 \\
&= 2^3 \times 3^1 \times 5^2 \times 7^3
\end{aligned}$$

Note also that $35k = 5^1 \times 7^1 \times k$.
Let d be the greatest common divisor of 205 800 and $35k$.
We want to find the number of possible values of $k \leq 2940$ for which d is a perfect square.
Since both 5 and 7 are prime divisors of 205 800 and $35k$, then 5 and 7 are both prime divisors of d (F5).
For d to be a perfect square, 5 and 7 must both divide d an even number of times (F4).
Since the prime powers of 5 and 7 in the prime factorization of 205 800 are 5^2 and 7^3, respectively, then for d to be a perfect square, it must be the case that 5^2 and 7^2 are factors of d.
Since $d = 5 \times 7 \times k$, then $k = 5 \times 7 \times j = 35j$ for some positive integer j.
Since $k \leq 2940$, then $35j \leq 2940$ which gives $j \leq 84$.
We now know that d is the gcd of $2^3 \times 3^1 \times 5^2 \times 7^3$ and $5^2 \times 7^2 \times j$.
What further information does this give us about j?

- j cannot be divisible by 3, otherwise d would have a factor of 3^1 (since both 205 800 and $35k$ would be divisible by 3) and cannot have a factor of 3^2 (since 205 800 does not) which would mean that d is not a perfect square.
- j cannot be divisible by 7, otherwise d has a factor of 7^3 and no larger power of 7, in which case d would not be a perfect square.
- If j is divisible by 2, then the prime factorization of j must include 2^2. In other words, the prime factorization of j cannot include 2^1 or 2^3.
- j can be divisible by 5 since even if j is divisible by 5, the power of 5 in d is already limited by the power of 5 in 205 800.
- j can be divisible by prime numbers other than 2, 3, 5 or 7 since 205 800 is not and so the gcd will not be affected.

Finally, we consider two cases: j is divisible by 2^2 but not by a larger power of 2, and j is not divisible by 2.

Case 1: j is divisible by 2^2 but not by a larger power of 2
Here, $j = 2^2h = 4h$ for some odd positive integer h.
Since $j \leq 84$, then $4h \leq 84$ which means that $h \leq 21$.
Knowing that j cannot be divisible by 3 or by 7, this means that the possible values of h are $1, 5, 11, 13, 17, 19$.
Each of these values of h produces a value of j that satisfies the conditions in the five bullets above.
There are thus 6 values of j in this case.

Case 2: j is not divisible by 2
Here, j is odd.
Knowing that j cannot be divisible by 3 or by 7 and that $j \leq 84$, this means that the possible values of j are:

$$1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 53, 55, 59, 61, 65, 67, 71, 73, 79, 83$$

There are thus 24 values of j in this case.

In total, there are 30 values of j and so there are 30 possible values of $k \leq 2940$ for which $(205\,800, 35k)$ is a happy pair.

Answer: (D)