The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Fryer Contest

Wednesday, April 15, 2020
(in North America and South America)

Thursday, April 16, 2020
(outside of North America and South America)

Solutions

1. (a) Locating point C on the graph, Cao pays \$7.00 for 14 posters which is a price of $\$7 \div 14 = \$0.50$ per poster.

   (b) From part (a), Daniel pays \$1.60 per poster and Cao pays \$0.50 per poster.
   Calculating, Annie pays \$10.00 for 5 posters or $\$10 \div 5 = \$2.00$ per poster, Bogdan pays $\$8.00 \div 8 = \$1.00$ per poster, and Emily pays $\$15.00 \div 15 = \$1.00$ per poster.
   Thus, Bogdan and Emily are paying the same price per poster.
   (Alternatively, we may have noticed that the line from the origin $(0, 0)$ passing through E also passes through B.
   The slope of this line represents the price in dollars per poster for Bogdan and for Emily.)

   (c) From part (a), Daniel paid the company that printed his first batch \$1.60 per poster.
   To print his second batch at the local library, Daniel would pay $\$60.00 \div 40 = \$1.50$ per poster.
   To spend less money, Daniel should print his second batch at the library.

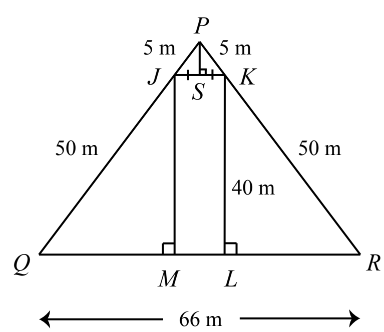
   Alternatively, Daniel paid \$16.00 for his first batch of 10 posters and thus would pay $4 \times \$16.00 = \$64.00$ to print his second batch of 40 posters using the same company that printed his first batch.
   Since the cost to print the 40 posters at the library is \$60.00, he should print them at the library to save money.

   (d) In part (b), we calculated that Emily's printing company charges her \$1.00 per poster.
   Since this is a fixed price per poster, then Emily paid \$1.00 per poster for each of the 25 posters that she had printed, for a total of \$25.00 spent.
   Annie and Emily each printed 25 posters and spent the same amount of money, \$25.00.
   Annie paid \$10.00 to print her first 5 posters, and so she spent $\$25.00 - \$10.00 = \$15.00$ to have her additional $25 - 5 = 20$ posters printed.
   Thus, Annie was charged $\$15.00 \div 20 = \$0.75$ per additional poster.

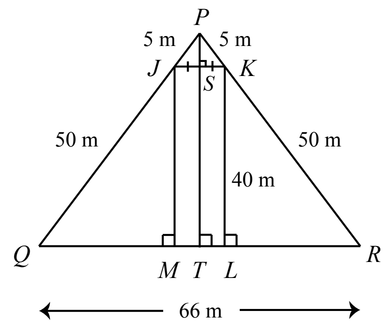
2. (a) In $\triangle KLR$, we have $\angle KLR = 90^\circ$ and by using the Pythagorean Theorem, we get $LR^2 = 50^2 - 40^2 = 900$ and so $LR = \sqrt{900} = 30$ m (since $LR > 0$).

   (b) We begin by showing that $\triangle JMQ$ is congruent to $\triangle KLR$.
   Since $JKLM$ is a rectangle, then $JM = KL = 40$ m.
   In addition, hypotenuse JQ has the same length as hypotenuse KR, and so $\triangle JMQ$ is congruent to $\triangle KLR$ by HS congruence.
   Thus, $MQ = LR = 30$ m and so $ML = 66 - 30 - 30 = 6$ m.

   (c) Since $PJ = PK = 5$ m, $\triangle PJK$ is isosceles and so the height, PS, drawn from P to JK bisects JK, as shown.
   Since $JKLM$ is a rectangle, then $JK = ML = 6$ m and so $SK = \frac{JK}{2} = 3$ m.
   Using the Pythagorean Theorem in $\triangle PSK$, we get $PS^2 = 5^2 - 3^2 = 16$ and so $PS = 4$ m (since $PS > 0$).
   Thus the height of $\triangle PJK$ drawn from P to JK is 4 m.



(d) We begin by determining the area of $\triangle PQR$.
Construct the height of $\triangle PQR$ drawn from P to T on QR, as shown.
Since PT is perpendicular to QR, then PT is parallel to KL (since KL is also perpendicular to QR).
By symmetry, PT passes through S, and so the height PT is equal to $PS + ST = PS + KL$ or $4 + 40 = 44$ m.
The area of $\triangle PQR$ is $\frac{1}{2} \times QR \times PT = \frac{1}{2} \times 66 \times 44 = 1452 \text{ m}^2$.
The area of $JKLM$ is $ML \times KL = 6 \times 40 = 240 \text{ m}^2$.
The fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$ is $\frac{240}{1452} = \frac{20}{121}$.



3. (a) If the 5^{th} term in a Dlin sequence is 142, then the 6^{th} term is $(142+1) \times 2 = 143 \times 2 = 286$.
To determine the 4^{th} term in the sequence given the 5^{th}, we "undo" adding 1 followed by doubling the result by first dividing the 5^{th} term by 2 and then subtracting 1 from the result.
To see this, consider that if two consecutive terms in a Dlin sequence are a followed by b, then $b = (a + 1) \times 2$.
To determine the operations needed to find a given b (that is, to move backward in the sequence), we rearrange this equation to solve for a.

$$\begin{aligned} b &= (a+1) \times 2 \\ \tfrac{b}{2} &= a + 1 \\ \tfrac{b}{2} - 1 &= a \end{aligned}$$

Thus if the 5^{th} term in the sequence is 142, then the 4^{th} term is $\frac{142}{2} - 1 = 71 - 1 = 70$.
(We may check that the term following 70 is indeed $(70 + 1) \times 2 = 142$.)

(b) If the 1^{st} term is 1406, then clearly this is a Dlin sequence that includes 1406.
If the 2^{nd} term in a Dlin sequence is 1406, then the 1^{st} term in the sequence is $\frac{1406}{2} - 1 = 703 - 1 = 702$.
If the 3^{rd} term in a Dlin sequence is 1406, then the 2^{nd} term is 702 (as calculated in the line above) and the 1^{st} term in the sequence is $\frac{702}{2} - 1 = 351 - 1 = 350$.
If the 4^{th} term in a Dlin sequence is 1406, then the 3^{rd} term is 702, the 2^{nd} term is 350, and the 1^{st} term in the sequence is $\frac{350}{2} - 1 = 175 - 1 = 174$.
At this point, we see that 174, 350, 702, and 1406 are possible 1^{st} terms which give a Dlin sequence that includes 1406.
We may continue this process of working backward (dividing by 2 and subtracting 1) to determine all possible 1^{st} terms which give a Dlin sequence that includes 1406.

$$1406 \to 702 \to 350 \to 174 \to \tfrac{174}{2} - 1 = 86 \to \tfrac{86}{2} - 1 = 42 \to \tfrac{42}{2} - 1 = 20 \to \tfrac{20}{2} - 1 = 9$$

Attempting to continue the process beyond 9 gives $\frac{9}{2} - 1 = \frac{7}{2}$ which is not possible since the 1^{st} term in a Dlin sequence must be a positive integer (and so all terms are positive integers).
Thus, the possible 1^{st} terms which give a Dlin sequence that includes 1406 are 9, 20, 42, 86, 174, 350, 702, and 1406.

(c) Each of the integers from 10 to 19 inclusive is a possible first term, and so we must determine the ones digit of each term which follows each of these ten possible first terms.
If the 1^{st} term is 10, then the 2^{nd} term $(10+1) \times 2 = 22$ has ones digit 2, and the 3^{rd} term

$(22+1)\times 2 = 46$ has ones digit 6.
If the 1^{st} term is 11, then the ones digit of the 2^{nd} term $(11+1)\times 2 = 24$ is 4, and the 3^{rd} term $(24+1)\times 2 = 50$ has ones digit 0.
Given each of the possible first terms, we list the ones digits of the 2^{nd} and 3^{rd} terms in the table below.

1^{st} term	10	11	12	13	14	15	16	17	18	19
Units digit of the 2^{nd} term	2	4	6	8	0	2	4	6	8	0
Units digit of the 3^{rd} term	6	0	4	8	2	6	0	4	8	2

From the table above, we see that the only ones digit which repeats itself is 8.
Thus, if the 1^{st} term in the sequence is 18 (has ones digit 8), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8 and so all terms will have the same ones digit, 8.
Similarly, if the 1^{st} term in the sequence is 13 (has ones digit 3), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8.
It then follows that all further terms after the first will have ones digit 8.
The 1^{st} terms (from 10 to 19 inclusive) which produce a Dlin sequence in which all terms after the 1^{st} term have the same ones digit are 13 and 18.

(d) If the 1^{st} term in a Dlin sequence is x, then the 2^{nd} term is $(x+1)\times 2 = 2x+2$, and the 3^{rd} term is $(2x+2+1)\times 2 = (2x+3)\times 2 = 4x+6$.
For example, if $x = 1$ (note that this is the smallest possible 1^{st} term of a Dlin sequence), then the 3^{rd} term is $4\times 1+6 = 10$, and if $x = 2$, the 3^{rd} term is $4\times 2+6 = 14$.
What is the largest possible value of x (the 1^{st} term of the sequence) which makes $4x+6$ (the 3^{rd} term of the sequence) less than or equal to 2020?
Setting $4x+6$ equal to 2020 and solving, we get $4x = 2014$ and so $x = 503.5$.
Since the 1^{st} term of the sequence must be a positive integer, the 3^{rd} term cannot be 2020.
Similarly, solving $4x+6 = 2019$, we get that x is not an integer and so the 3^{rd} term of a Dlin sequence cannot equal 2019.
When $4x+6 = 2018$, we get $4x = 2012$ and so $x = 503$.
Thus, if a Dlin sequence has 1^{st} term equal to 503, then the 3^{rd} term of the sequence is a positive integer between 1 and 2020, namely 2018.
Further, 503 is the largest possible 1^{st} term for which the 3^{rd} term has this property.
Each 1^{st} term x will give a different 3^{rd} term, $4x+6$.
Thus, to count the number of positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence, we may count the number of 1^{st} terms which give a 3^{rd} term having this property.
The smallest possible 1^{st} term is 1 (giving a 3^{rd} term of 10) and the largest possible 1^{st} term is 503 (which gives a 3^{rd} term of 2018).
Further, every value of x between 1 and 503 gives a different 3^{rd} term between 10 and 2018.
Thus, there are 503 positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence.

4. (a) There are 10 different ways to colour a 5×1 grid so that exactly 3 cells are red and 2 cells are blue.
One way to see this is to count the number of different ways to colour 2 cells blue, since each of the 3 remaining cells must be coloured red.
(Alternatively, we could count the number of different ways to colour 3 cells red.)
There are 5 cells that may be chosen first to be coloured blue.
After this chosen cell is coloured, there are 4 remaining cells that can be chosen to be

coloured blue. Thus, there are $5 \times 4 = 20$ such choices of cells to be coloured blue.
However, since the two blue cells are indistinguishable from one another, we have counted each of the different ways of colouring 2 cells blue twice.
For example, choosing to first colour the cell in the second row blue and then colouring the cell in the fifth row blue gives the same colouring as first choosing to colour the cell in the fifth row blue and then colouring the cell in the second row blue.
Thus, there are $20 \div 2 = 10$ different ways that a 5×1 grid can be coloured so that exactly 3 cells are red and 2 cells are blue. These 10 colourings are shown to the right.

B	B	B	B	R
B	R	R	R	B
R	B	R	R	B
R	R	B	R	R
R	R	R	B	R

R	R	R	R	R
B	B	R	R	R
R	R	B	B	R
B	R	B	R	B
R	B	R	B	B

(b) We begin by observing that there are exactly 2 different ways to colour each of the 13 cells (red or blue), and so there are a total of 2^{13} possible colourings of a 1×13 grid.
For each 1×13 grid, Carrie counts the number of cells coloured red, call this number r, and she counts the number of cells coloured blue, call this number b.
Since there are 13 cells in total, then $r + b = 13$ and either $r > b$ or $b > r$ (r and b cannot be equal since they are integers and their sum is odd).
If $r > b$, then $r > 13 - r$ which gives $2r > 13$ or $r > 6.5$ and so $r \geq 7$ (since r is an integer).
If $b > r$, then $b > 13 - b$ which gives $2b > 13$ or $b > 6.5$ and so $b \geq 7$ (since b is an integer).
Thus, if the number of cells coloured red is greater than the number of cells coloured blue, then Carrie writes down the number of red cells (since she writes the maximum of r and b), and this number is at least 7.
Alternatively, if the number of cells coloured blue is greater than the number of cells coloured red, then Carrie writes down the number of blue cells, and this number is at least 7.
In either case, each number that Carrie writes down in her list is between 7 and 13, inclusive.
At least one of the possible 2^{13} different colourings of a 1×13 grid has 7 cells coloured red and 6 cells coloured blue, and so Carrie's list includes at least one 7.
Since Carrie's list includes a 7 and every number in her list is 7 or greater, then the smallest number in her list is 7.

(c) In a $3 \times n$ grid, each column contains exactly 3 cells.
Each of these 3 cells can be coloured in one of two ways, either red or blue.
Since there are two ways that each of the 3 cells may be coloured, each column in a $3 \times n$ grid can be coloured in one of $2^3 = 8$ different ways.
Thus, it is possible to colour a $3 \times n$ grid for integers n, $1 \leq n \leq 8$, so that each column is coloured in a different way (such an example of a 3×8 grid is shown).

R	R	R	B	R	B	B	B
R	R	B	R	B	R	B	B
R	B	R	R	B	B	R	B

Since there are only 8 different ways to colour the 3 cells in any column, then every colouring of a 3×9 grid must have at least two columns that are coloured in an identical way.
Thus, the smallest possible value of n is 9.

(d) The given statement is true.
In a 5×41 grid, there are 5 rows and so each of the 41 columns has 5 cells.
In each column, at least 3 of the 5 cells must be the same colour.
In each column, either there are more cells coloured red than blue or there are more cells coloured blue than red (they can't be equal in number since 5 is odd).

In each column, if there are more cells coloured red than blue, we call that column a red column.
Alternatively, if there are more cells coloured blue than red, we call that column a blue column.
Let the total number of red columns be $\mathbb{R}$ and the number of blue columns be $\mathbb{B}$, and so $\mathbb{R} + \mathbb{B} = 41$.
Similar to the argument in part (b), if $\mathbb{R} > \mathbb{B}$, then $\mathbb{R} \geq 21$, otherwise $\mathbb{B} \geq 21$.
Assume that $\mathbb{R} \geq 21$ (the argument that follows can be made in a similar way if $\mathbb{B} \geq 21$).
In this case, each of these 21 (or more) columns has more cells coloured red than those that are coloured blue and so each of these columns has at least 3 cells coloured red.
We will show that the location of 3 cells coloured red is the same in at least 3 of 21 columns.
Of all red columns, consider the first 21 (there are at least 21).
Moving from top to bottom within each of these red columns, consider the first 3 red cells (there are at least 3 cells coloured red).
What is the maximum number of ways to colour each column so that exactly 3 cells are coloured red?
Since a 5×1 grid is the size of each column in a 5×41 grid, this is the question that was answered in part (a), and so there are 10 such ways.
In the 21 red columns, assume that at most 2 columns have the same 3 cells coloured red.
Since there are only 10 different ways to colour each column so that exactly 3 cells are coloured red, then there are at most $2 \times 10 = 20$ such columns.
However, there are 21 red columns and so we have a contradiction.
Thus, our assumption that in the 21 red columns, there are at most 2 columns in which the same 3 cells are coloured red was incorrect, and so there must be at least 3 columns which have the same 3 cells that are coloured red.
The 9 cells located at the intersection of these 3 columns and the 3 rows containing the red cells within these columns are all coloured red.