The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Fermat Contest

(Grade 11)

Tuesday, February 25, 2020
(in North America and South America)

Wednesday, February 26, 2020
(outside of North America and South America)

Solutions

1. Since $OPQR$ is a rectangle with two sides on the axes, then its sides are horizontal and vertical. Since PQ is horizontal, the y-coordinate of Q is the same as the y-coordinate of P, which is 3. Since QR is vertical, the x-coordinate of Q is the same as the x-coordinate of R, which is 5. Therefore, the coordinates of Q are $(5, 3)$.

   ANSWER: (B)

2. Calculating,

$$3 \times 2020 + 2 \times 2020 - 4 \times 2020 = 2020 \times (3 + 2 - 4) = 2020 \times 1 = 2020$$

   Alternatively,

$$3 \times 2020 + 2 \times 2020 - 4 \times 2020 = 6060 + 4040 - 8080 = 10\,100 - 8080 = 2020$$

   ANSWER: (E)

3. Expanding and simplifying, $(x + 1)^2 - x^2 = (x^2 + 2x + 1) - x^2 = 2x + 1$.

   ANSWER: (A)

4. Ewan's sequence starts with 3 and each following number is 11 larger than the previous number. Since every number in the sequence is some number of 11s more than 3, this means that each number in the sequence is 3 more than a multiple of 11. Furthermore, every such positive integer is in Ewan's sequence.
   Since $110 = 11 \times 10$ is a multiple of 11, then $113 = 110 + 3$ is 3 more than a multiple of 11, and so is in Ewan's sequence.
   Alternatively, we could write Ewan's sequence out until we get into the correct range:

$$3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, \ldots$$

   ANSWER: (A)

5. Calculating, $\sqrt{\dfrac{\sqrt{81} + \sqrt{81}}{2}} = \sqrt{\dfrac{9 + 9}{2}} = \sqrt{9} = 3$.

   ANSWER: (A)

6. Since 12 and 21 are multiples of 3 ($12 = 4 \times 3$ and $21 = 7 \times 3$), the answer is not (A) or (D).
   16 is a perfect square ($16 = 4 \times 4$) so the answer is not (C).
   The sum of the digits of 26 is 8, which is not a prime number, so the answer is not (E).
   Since 14 is not a multiple of a three, 14 is not a perfect square, and the sum of the digits of 14 is $1 + 4 = 5$ which is prime, then the answer is (B) 14.

   ANSWER: (B)

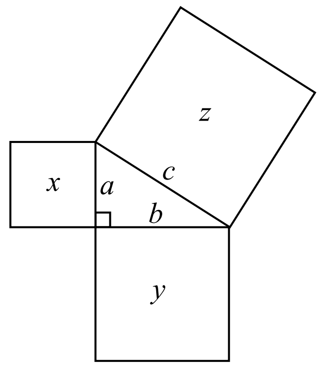
7. Since WXY is a straight angle, then $p^\circ + q^\circ + r^\circ + s^\circ + t^\circ = 180^\circ$ and so $p + q + r + s + t = 180$.
   To calculate the average of p, q, r, s, and t, we add the five numbers and divide by 5.
   Therefore, the average of p, q, r, s, and t is $\dfrac{p + q + r + s + t}{5} = \dfrac{180}{5} = 36$.

   ANSWER: (B)

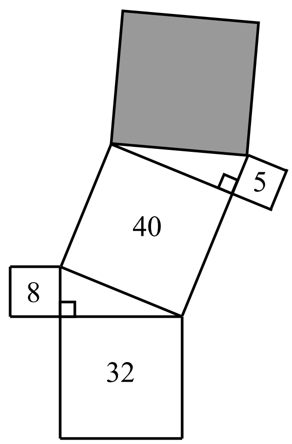
8. Since $8 = 2 \times 2 \times 2 = 2^3$, then $8^{20} = (2^3)^{20} = 2^{3 \times 20} = 2^{60}$.
   Thus, if $2^n = 8^{20}$, then $n = 60$.

   ANSWER: (B)

9. The Pythagorean Theorem tells us that if a right-angled triangle has sides of length a, b and c, with c the hypotenuse, then $a^2 + b^2 = c^2$.
Since the area of a square of side length a is a^2, the Pythagorean Theorem can be re-phrased to say that the sum of the areas of the squares that can be drawn on the two shorter sides equals the area of the square that can be drawn on the hypotenuse. (In the figure below, this says that $x + y = z$ where x, y and z are the areas of the squares, as shown.)



In the given diagram, this means that the area of the unmarked, unshaded square is $8+32 = 40$.



This means that the area of the shaded square is $40 + 5 = 45$.

Answer: (B)

10. We are given that s and t are positive integers and that $s(s - t) = 29$.
Since s and t are positive, then $s - t$ is less than s.
Since s is positive and 29 is positive and $s(s - t) = 29$, then $s - t$ must also be positive.
Since 29 is a prime number, the only way that it can be written as a product of two positive integers is $29 = 29 \cdot 1$.
Since $s(s - t) = 29$ and $s > s - t$, then we must have $s = 29$ and $s - t = 1$.
Since $s = 29$ and $s - t = 1$, we obtain $t = 28$.
Therefore, $s + t = 29 + 28 = 57$.

Answer: (C)

11. Each of the first and second columns has 4 X's in it, which means that at least 2 X's need to be moved. We will now show that this can be actually done by moving 2 X's.
Each of the first and second rows has 4 X's in it, so we move the two X's on the main diagonals, since this will remove X's from the first and second columns and the first and second rows simultaneously.
The fifth column starts with one X in it, so we move the two X's to the fifth column into the rows that only contain 2 X's. Doing this, we obtain:

O	X	X	X	
X	O	X		X
X	X			X*
X	X		X	
		X	X	X*

(The cells from which X's have been removed are marked with O's; the cells to which X's are moved are marked with X*'s.)
Therefore, the smallest number of X's that must be moved is 2.

Answer: (B)

12. Since Harriet ran 720 m at 3 m/s, then this segment took her $\dfrac{720 \text{ m}}{3 \text{ m/s}} = 240$ s.
In total, Harriet ran 1000 m in 380 s, so the remaining part of the course was a distance of 1000 m − 720 m = 280 m which she ran in 380 s − 240 s = 140 s.
Since she ran this section at a constant speed of v m/s, then $\dfrac{280 \text{ m}}{140 \text{ s}} = v$ m/s which means that $v = 2$.

Answer: (A)

13. Since the sum of any two adjacent numbers is constant, then $2 + x = x + y$.
This means that $y = 2$ and makes the list $2, x, 2, 5$.
This means that the sum of any two adjacent numbers is $2 + 5 = 7$, and so $x = 5$.
Therefore, $x - y = 5 - 2 = 3$.

Answer: (C)

14. If $\frac{2}{7}$ of the roses are to be yellow, then the remaining $\frac{5}{7}$ of the roses are to be red.
Since there are 30 red roses and these are to be $\frac{5}{7}$ of the roses, then $\frac{1}{7}$ of the total number of roses would be $30 \div 5 = 6$, which means that there would be $6 \times 7 = 42$ roses in total.
If there are 42 roses of which 30 are red and the rest are yellow, then there are $42 - 30 = 12$ yellow roses.
Since there are 19 yellow roses to begin, then $19 - 12 = 7$ yellow roses are removed.

Answer: (E)

15. When $N = 3x + 4y + 5z$ with each of x, y and z equal to either 1 or -1, there are 8 possible combinations of values for x, y and z:

x	y	z	N
1	1	1	12
1	1	-1	2
1	-1	1	4
1	-1	-1	-6
-1	1	1	6
-1	1	-1	-4
-1	-1	1	-2
-1	-1	-1	-12

From this information, N cannot equal 0, N is never odd, N can equal 4, and N is always even. Therefore, exactly one of the four given statements is true.

Answer: (B)

16. We note that $\dfrac{x+y}{x} = \dfrac{x}{x} + \dfrac{y}{x} = 1 + \dfrac{y}{x}$.
The greatest possible value of $\dfrac{x+y}{x} = 1 + \dfrac{y}{x}$ thus occurs when $\dfrac{y}{x}$ is as great as possible.
Since x is always negative and y is always positive, then $\dfrac{y}{x}$ is negative.
Therefore, for $\dfrac{y}{x}$ to be as great as possible, it is as least negative as possible (i.e. closest to 0 as possible).
Since x is negative and y is positive, this happens when x is as negative as possible and y is as small as possible – that is, when $x = -4$ and $y = 2$.
Therefore, the greatest possible value of $\dfrac{x+y}{x}$ is $1 + \dfrac{2}{-4} = \dfrac{1}{2}$.

Answer: (E)

17. Since $\triangle PQR$ is right-angled at Q, its area equals $\frac{1}{2} \cdot PQ \cdot QR$.
Since its area is 30 and $PQ = 5$, then $\frac{1}{2} \cdot 5 \cdot QR = 30$ and so $QR = 30 \cdot \frac{2}{5} = 12$.
By the Pythagorean Theorem, we know that

$$PR^2 = PQ^2 + QR^2 = 5^2 + 12^2 = 25 + 144 = 169$$

Since $PR > 0$, then $PR = \sqrt{169} = 13$.
If we now consider $\triangle PQR$ as having base PR and perpendicular height QS, we see that its area equals $\frac{1}{2} \cdot PR \cdot QS$.
Since its area is 30 and $PR = 13$, then $\frac{1}{2} \cdot 13 \cdot QS = 30$ which gives $QS = 30 \cdot \frac{2}{13} = \frac{60}{13}$.

Answer: (A)

18. Suppose that the four teams in the league are called W, X, Y, and Z.
Then there is a total of 6 games played:

W against X, W against Y, W against Z, X against Y, X against Z, Y against Z

In each game that is played, either one team is awarded 3 points for a win and the other is awarded 0 points for a loss (for a total of 3 points between the two teams), or each team is awarded 1 point for a tie (for a total of 2 points between the two teams).
Since 6 games are played, then the theoretical maximum number of points that could be awarded is $6 \cdot 3 = 18$ and the theoretical minimum number of points that can be awarded is $6 \cdot 2 = 12$.
In particular, this means that it is not possible for the total number of points to be 11.
We can show that each of the possibilities from 12 to 18 points, inclusive, is actually possible.
Therefore, S cannot equal 11.

Answer: (C)

19. When $(3 + 2x + x^2)(1 + mx + m^2x^2)$ is expanded, the terms that include an x^2 will come from multiplying a constant with a term that includes x^2 or multiplying two terms that includes x.
In other words, the term that includes x^2 will be

$$3 \cdot m^2x^2 + 2x \cdot mx + x^2 \cdot 1 = 3m^2x^2 + 2mx^2 + x^2 = (3m^2 + 2m + 1)x^2$$

From the condition that the coefficient of this term equals 1, we see that $3m^2 + 2m + 1 = 1$ which gives $3m^2 + 2m = 0$ or $m(3m + 2) = 0$, which means that $m = 0$ or $m = -\frac{2}{3}$.
The sum of these possible values of m is $-\frac{2}{3}$.

Answer: (B)

20. When a dot is removed from a face with an even number of dots, that face then has an odd number of dots.
When a dot is removed from a face with an odd number of dots, that face then has an even number of dots.
Initially, there are 3 faces with an even number of dots and 3 faces with an odd number of dots.
If a dot is removed from a face with an even number of dots, there are then 4 faces with an odd number of dots and 2 faces with an even number of dots. This means that the probability of rolling an odd number after a dot is removed is $\frac{4}{6}$ in this case.
If a dot is removed from a face with an odd number of dots, there are then 2 faces with an odd number of dots and 4 faces with an even number of dots. This means that the probability of rolling an odd number after a dot is removed is $\frac{2}{6}$ in this case.
Since there are $2 + 3 + 4 + 5 + 6 + 7 = 27$ dots on the faces, then the probability that a dot is removed from the face with 2 dots is $\frac{2}{27}$, from the face with 3 dots is $\frac{3}{27}$, and so on.
Thus, the probability that a dot is removed from the face with 2 dots and then an odd number is rolled is the product of the probabilities, which is $\frac{2}{27} \cdot \frac{2}{3}$, since there are now 4 odd faces and 2 even faces.
Similarly, the probability that a dot is removed from the face with 3 dots and then an odd number is rolled is $\frac{3}{27} \cdot \frac{1}{3}$.
Continuing in this way, the probability of rolling an odd number after a dot is removed is $\frac{2}{27} \cdot \frac{2}{3} + \frac{3}{27} \cdot \frac{1}{3} + \frac{4}{27} \cdot \frac{2}{3} + \frac{5}{27} \cdot \frac{1}{3} + \frac{6}{27} \cdot \frac{2}{3} + \frac{7}{27} \cdot \frac{1}{3}$.
This equals $\frac{2}{3} \cdot (\frac{2}{27} + \frac{4}{27} + \frac{6}{27}) + \frac{1}{3}(\frac{3}{27} + \frac{5}{27} + \frac{7}{27}) = \frac{2}{3} \cdot \frac{12}{27} + \frac{1}{3} \cdot \frac{15}{27} = \frac{8}{27} + \frac{5}{27} = \frac{13}{27}$.

Answer: (C)

21. If the product of three numbers x, 36 and y is 2592, then $x \cdot 36 \cdot y = 2592$ and so $xy = \dfrac{2592}{36} = 72$.
If x and y are positive integers with $xy = 72$, then we have the following possibilities:

x	y	$x+y$
72	1	73
36	2	38
24	3	27
18	4	22
12	6	18
9	8	17

We have assumed that $x > y$ since we have not assigned an order to x, 36 and y.
In the given problem, we want to put four pairs of numbers in the outer circles so that the 9 numbers are different and the sum of the 9 numbers is as large as possible.
Putting this another way, we want to choose 4 of the 6 pairs in the table above (knowing that we cannot choose the pair 36 and 2 since 36 is already in the middle circle) to make the sum as large as possible.
Since we know the sums of the pairs, we choose the pairs with the four largest sums.
This means that the sum of the 9 numbers will be $(72+1)+(24+3)+(18+4)+(12+6)+36$ which equals $73+27+22+18+36$ or 176.

Answer: (B)

22. Since $x^2+3xy+y^2 = 909$ and $3x^2+xy+3y^2 = 1287$, then

$$\begin{aligned}(x^2+3xy+y^2)+(3x^2+xy+3y^2) &= 909+1287\\ 4x^2+4xy+4y^2 &= 2196\\ x^2+xy+y^2 &= 549\end{aligned}$$

Since $x^2+3xy+y^2 = 909$ and $x^2+xy+y^2 = 549$, then

$$\begin{aligned}(x^2+3xy+y^2)-(x^2+xy+y^2) &= 909-549\\ 2xy &= 360\\ xy &= 180\end{aligned}$$

Since $x^2+3xy+y^2 = 909$ and $xy = 180$, then

$$\begin{aligned}(x^2+3xy+y^2)-xy &= 909-180\\ x^2+2xy+y^2 &= 729\\ (x+y)^2 &= 27^2\end{aligned}$$

Therefore, $x+y = 27$ or $x+y = -27$. This also shows that $x+y$ cannot equal any of 39, 29, 92, and 41.
(We can in fact solve the system of equations $x+y = 27$ and $xy = 180$ for x and y to show that there do exist real numbers x and y that are solutions to the original system of equations.)
Therefore, a possible value for $x+y$ is (A) 27.

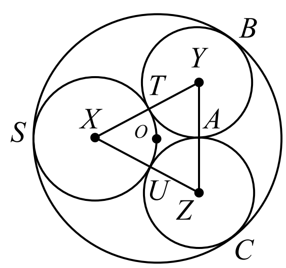
Answer: (A)

23. *Solution 1*
Since $f(x) = ax + b$ for all real numbers x, then $f(t) = at + b$ for some real number t.
When $t = bx + a$, we obtain $f(bx + a) = a(bx + a) + b = abx + (a^2 + b)$.
We also know that $f(bx + a) = x$ for all real numbers x.
This means that $abx + (a^2 + b) = x$ for all real numbers x and so $(ab - 1)x + (a^2 + b) = 0$ for all real numbers x.
For this to be true, it must be the case that $ab = 1$ and $a^2 + b = 0$.
From the second equation $b = -a^2$ which gives $a(-a^2) = 1$ and so $a^3 = -1$, which means that $a = -1$.
Since $b = -a^2$, then $b = -1$ as well, which gives $a + b = -2$.

Solution 2
Since $f(x) = ax + b$ for all x, then when $x = a$, we obtain $f(a) = a^2 + b$.
Since $f(bx + a) = x$ for all x, then when $x = 0$, we obtain $f(a) = 0$.
Comparing values for $f(a)$, we obtain $a^2 + b = 0$ or $b = -a^2$.
This gives $f(x) = ax - a^2$ for all real numbers x and $f(-a^2x + a) = x$ for all real numbers x.
Since $f(-a^2x + a) = x$ for all x, then when $x = -1$, we obtain $f(a^2 + a) = -1$.
Since $f(x) = ax - a^2$ for all x, then when $x = a^2 + a$, we obtain $f(a^2 + a) = a(a^2 + a) - a^2$.
Comparing values for $f(a^2 + a)$, we obtain $a(a^2 + a) - a^2 = -1$ or $a^3 = -1$.
Since a is a real number, then $a = -1$.
Since $b = -a^2$, then $b = -1$, which gives $a + b = -2$.
Checking, we see that if $f(x) = -x - 1$, then $f(-x - 1) = -(-x - 1) - 1 = x$, as required.

ANSWER: (E)

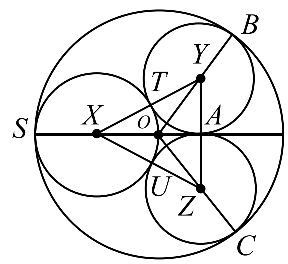
24. Suppose the centre of the largest circle is O.
Suppose that the circle with centre X touches the largest circle at S and the two circles with centres Y and Z at T and U, respectively.
Suppose that the circles with centres Y and Z touch each other at A, and the largest circle at B and C, respectively.
Join X to Y, X to Z, and Y to Z.



(Note that the diagram has been re-drawn here so that the circle with centre X actually appears to pass through the centre of the largest circle.)
Since the circles are tangent at points T and U, line segments XY and XZ pass through T and U, respectively.
Further, $XY = XT + TY = 1 + r$, since the circles with centres X and Y have radii 1 and r, respectively.
Similarly, $XZ = 1 + r$.
Also, $YA = ZA = YB = ZC = r$, since these are radii of the two circles.
When one circle is inside another circle, and the two circles touch at a point, then the radii of the two circles that pass through this point lie on top of each other. This is because the

circles have a common tangent at the point where they touch and this common tangent will be perpendicular to each of the radii.
Since the circle with centre X touches the largest circle at S, then X lies on OS.
In the largest circle, consider the diameter that passes through X.
Since the circle with centre X passes through O, then the radius of the largest circle is twice that of the circle with centre X, or 2.
It is also the case that $XO = 1$.
Next, we join O to B. Since the circles with centres O and Y touch at B, then OB passes through Y. This means that $OY = OB - BY = 2 - r$. Similarly, $OZ = 2 - r$.
Further, by symmetry in the largest circle, the diameter through X also passes through A, the point at which the two smallest circles touch:

> To see this more formally, draw the common tangent through A to the circles with centres Y and Z.
> This line is perpendicular to YZ, since it is tangent to both circles.
> Since $\triangle OYZ$ is isosceles with $OY = OZ$, the altitude through the midpoint A of its base passes through O.
> Similarly, $\triangle XYZ$ is isosceles with $XY = XZ$ and so its altitude through A passes through X.
> Since the line perpendicular to YZ at A passes through both O and X, it is the diameter that passes through X.



Now, we consider $\triangle XYA$ and $\triangle OYA$, each of which is right-angled at A.
By the Pythagorean Theorem,

$$OA = \sqrt{OY^2 - YA^2} = \sqrt{(2-r)^2 - r^2} = \sqrt{4 - 4r + r^2 - r^2} = \sqrt{4 - 4r}$$

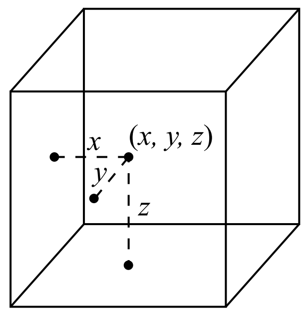
Again, using the Pythagorean Theorem,

$$\begin{aligned} XA^2 + YA^2 &= XY^2 \\ (XO + OA)^2 + r^2 &= (1 + r)^2 \\ (1 + \sqrt{4 - 4r})^2 &= 1 + 2r + r^2 - r^2 \\ 1 + 2\sqrt{4 - 4r} + (4 - 4r) &= 1 + 2r \\ 2\sqrt{4 - 4r} &= 6r - 4 \\ \sqrt{4 - 4r} &= 3r - 2 \\ 4 - 4r &= (3r - 2)^2 \qquad \text{(squaring both sides)} \\ 4 - 4r &= 9r^2 - 12r + 4 \\ 8r &= 9r^2 \end{aligned}$$

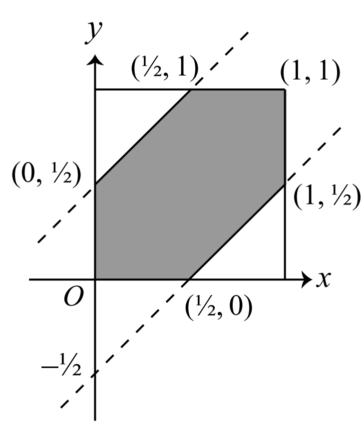
Since $r \neq 0$, then $9r = 8$ and so $r = \frac{8}{9} \approx 0.889$.
Of the given choices, this is closest to (E) 0.89.

Answer: (E)

25. Consider a $1 \times 1 \times 1$ cube.
We associate a triple (x, y, z) of real numbers with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and $0 \leq z \leq 1$ with a point inside this cube by letting x be the perpendicular distance of a point from the left face, y the perpendicular distance of a point from the front face, and z the perpendicular distance from the bottom face.
We call this point (x, y, z). Choosing x, y and z randomly and independently between 0 and 1 is equivalent to randomly and uniformly choosing a point (x, y, z) on or inside the cube.



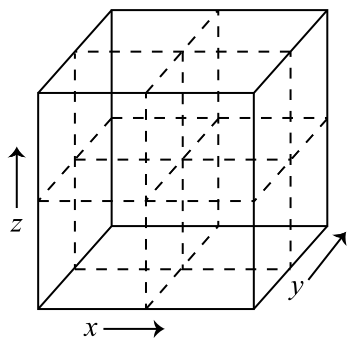
The conditions that $-\frac{1}{2} < x - y < \frac{1}{2}$ and $-\frac{1}{2} < x - z < \frac{1}{2}$ restrict the values of x, y and z that can be chosen, which translates into restricting the points inside the cube that satisfy these conditions. Hence, these restrictions determine a region inside this cube.
The probability that a point randomly chosen inside this cube satisfies the given conditions will be equal to the volume of the region defined by the conditions divided by the volume of the entire cube.
Since the volume of the cube is 1, then the probability will equal the volume of the region defined by those conditions.
Consider now the region in the xy-plane defined by $-\frac{1}{2} < x - y < \frac{1}{2}$.
Re-arranging these inequalities, we obtain $x - \frac{1}{2} < y < x + \frac{1}{2}$, which means that a point (x, y) that satisfies these conditions lies above the line with equation $y = x - \frac{1}{2}$ and below the line with equation $y = x + \frac{1}{2}$.
Restricting to $0 \leq x \leq 1$ and $0 \leq y \leq 1$, we obtain the region shown:



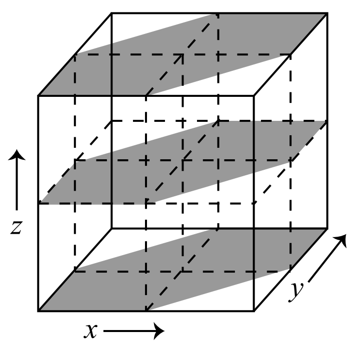
Since a point (x, y, z) in the region satisifes $-\frac{1}{2} < x - y < \frac{1}{2}$, these conditions allow us to "slice" the cube from above keeping the portion that looks like the region above. The points that remain are exactly those that satisfy this condition.
Similarly, the conditions $-\frac{1}{2} < x - z < \frac{1}{2}$ give $x - \frac{1}{2} < z < x + \frac{1}{2}$, which has the same shape in the xz-plane.

Therefore, we can slice the cube from front to back to look like this shape. Now, we need to determine the volume of the remaining region.

To determine the volume of the region, we split the $1 \times 1 \times 1$ cube into eight cubes each measuring $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$.



When this cube is sliced by the restrictions corresponding to $x - \frac{1}{2} < y < x + \frac{1}{2}$, the back left and front right cubes on the top and bottom layers are sliced in half.



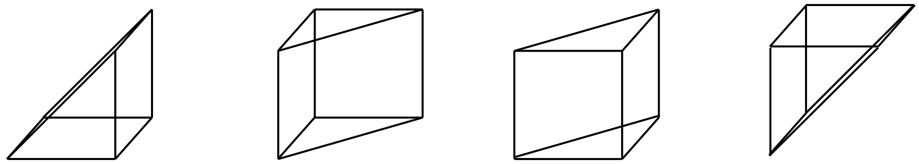
When this cube is sliced by the restrictions corresponding to $x - \frac{1}{2} < z < x + \frac{1}{2}$, the top left and bottom right cubes in the front and back are sliced in half.
The eight little cubes are sliced as follows:

Little cube	Sliced by $x - \frac{1}{2} < y < x + \frac{1}{2}$	Sliced by $x - \frac{1}{2} < z < x + \frac{1}{2}$
Bottom front left	No	No
Bottom front right	Yes	Yes
Bottom back left	Yes	No
Bottom back right	No	Yes
Top front left	No	Yes
Top front right	Yes	No
Top back left	Yes	Yes
Top back right	No	No

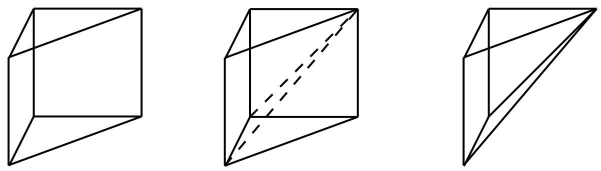
This means that we can consider the little cubes as follows:

- Bottom front left and top back right: these cubes are not sliced in either direction and so contribute $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to the volume of the solid.

- Bottom back left, bottom back right, top front left, top front right: these cubes are sliced in half in one direction and are not sliced in the other direction, and so contribute $\frac{1}{2}$ of their volume (or $\frac{1}{16}$ each) to the solid.



- Top back left and bottom front right: Each of these cubes is sliced in half in two directions. The first slice cuts the cube into a triangular prism, whose volume is half of the volume of the little cube, or $\frac{1}{16}$. The second slice creates a square-based pyramid out of this prism. The pyramid has base with edge length $\frac{1}{2}$ and height $\frac{1}{2}$, and so has volume $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{24}$.



Therefore, the volume of the solid is $2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + 2 \cdot \frac{1}{24} = \frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12}$.
Finally, this means that the required probability is $\frac{7}{12}$.

ANSWER: (B)