The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Cayley Contest

(Grade 10)

Tuesday, February 25, 2020
(in North America and South America)

Wednesday, February 26, 2020
(outside of North America and South America)

Solutions

1. Simplifying, $\dfrac{20-20}{20+20} = \dfrac{0}{40} = 0$.

   Answer: (A)

2. When $x = 3$ and $y = 4$, we get $xy - x = 3 \times 4 - 3 = 12 - 3 = 9$.
   Alternatively, $xy - x = x(y-1) = 3 \times 3 = 9$.

   Answer: (D)

3. Since $OPQR$ is a rectangle with two sides on the axes, then its sides are horizontal and vertical.
   Since PQ is horizontal, the y-coordinate of Q is the same as the y-coordinate of P, which is 3.
   Since QR is vertical, the x-coordinate of Q is the same as the x-coordinate of R, which is 5.
   Therefore, the coordinates of Q are $(5, 3)$.

   Answer: (B)

4. If $0 < a < 20$, then $\dfrac{1}{a} > \dfrac{1}{20}$. Therefore, $\dfrac{1}{15} > \dfrac{1}{20}$ and $\dfrac{1}{10} > \dfrac{1}{20}$.

   Also, $\dfrac{1}{20} = 0.05$ which is less than both 0.5 and 0.055.

   Lastly, $\dfrac{1}{20} > \dfrac{1}{25}$ since $0 < 20 < 25$.

   Therefore, $\dfrac{1}{25}$ is the only one of the choices that is less than $\dfrac{1}{20}$.

   Answer: (B)

5. Since QST is a straight angle, then $\angle QSP = 180^\circ - \angle TSP = 180^\circ - 50^\circ = 130^\circ$.
   Now $\angle RQS$ is an exterior angle for $\triangle QSP$.
   This means that $\angle RQS = \angle QSP + \angle SPQ$.
   Using the information that we know, $150^\circ = 130^\circ + x^\circ$ and so $x = 150 - 130 = 20$.
   (Alternatively, we could have noted that $\angle RQS$ and $\angle SQP$ are supplementary and then used the sum of the angles in $\triangle QSP$.)

   Answer: (E)

6. From the bar graph, Matilda saw 6 goldfinches, 9 sparrows, and 5 grackles.
   In total, she saw $6 + 9 + 5 = 20$ birds.
   This means that the percentage of birds that were goldfinches is $\dfrac{6}{20} \times 100\% = \dfrac{3}{10} \times 100\% = 30\%$.

   Answer: (C)

7. Since the average of m and n is 5, then $\dfrac{m+n}{2} = 5$ which means that $m + n = 10$.
   In order for n to be as large as possible, we need to make m as small as possible.
   Since m and n are positive integers, then the smallest possible value of m is 1, which means that the largest possible value of n is $n = 10 - m = 10 - 1 = 9$.

   Answer: (C)

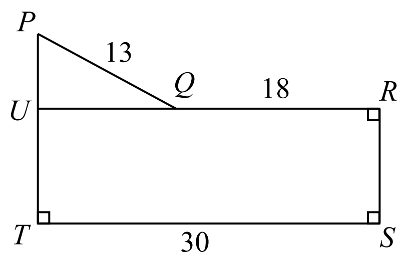
8. To determine 30% of Roman's \$200 prize, we calculate $\$200 \times 30\% = \$200 \times \frac{30}{100} = \$2 \times 30 = \$60$.
   After Roman gives \$60 to Jackie, he has $\$200 - \$60 = \$140$ remaining.
   He splits 15% of this between Dale and Natalia.
   The total that he splits is $\$140 \times 15\% = \$140 \times 0.15 = \$21$.
   Since Roman splits \$21 equally between Dale and Natalia, then Roman gives Dale a total of $\$21 \div 2 = \$10.50$.

   Answer: (A)

9. The 1st row has 0 shaded squares and 1 unshaded square.
The 2nd row has 1 shaded square and 2 unshaded squares.
The 3rd row has 2 shaded squares and 3 unshaded squares.
The 4th row has 3 shaded squares and 4 unshaded squares.
Because each row has 2 more squares than the previous row and the squares in each row alternate betweeen unshaded and shaded, then each row has exactly 1 more shaded square than the previous row.
This means that, moving from the 4th row to the 2020th row, a total of $2020 - 4 = 2016$ additional shaded squares are added.
Thus, the 2020th row has $3 + 2016 = 2019$ shaded squares.

ANSWER: (D)

10. We extend RQ to the left until it meets PT at point U, as shown.



Because quadrilateral $URST$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $UT = RS$ and $UR = TS = 30$.
Since $UR = 30$, then $UQ = UR - QR = 30 - 18 = 12$.
Now $\triangle PQU$ is right-angled at U.
By the Pythagorean Theorem, since $PU > 0$, we have

$$PU = \sqrt{PQ^2 - UQ^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$$

Since the perimeter of $PQRST$ is 82, then $13 + 18 + RS + 30 + (UT + 5) = 82$.
Since $RS = UT$, then $2 \times RS = 82 - 13 - 18 - 30 - 5 = 16$ and so $RS = 8$.
Finally, we can calculate the area of $PQRST$ by splitting it into $\triangle PQU$ and rectangle $URST$.
The area of $\triangle PQU$ is $\frac{1}{2} \times UQ \times PU = \frac{1}{2} \times 12 \times 5 = 30$.
The area of rectangle $URST$ is $RS \times TS = 8 \times 30 = 240$.
Therefore, the area of pentagon $PQRST$ is $30 + 240 = 270$.

ANSWER: (E)

11. Since

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

then

$$5 + 10 + 15 + \cdots + 40 + 45 = 5(1 + 2 + 3 + \cdots + 8 + 9) = 5(45) = 225$$

ANSWER: (A)

12. Suppose that the length, width and height of the prism are the positive integers a, b and c.
Since the volume of the prism is 21, then $abc = 21$.
We note that each of a, b and c is a positive divisor of 21.
The positive divisors of 21 are 1, 3, 7, and 21, and the only way to write 21 as a product of three different integers is $1 \times 3 \times 7 = 21$.
Therefore, the length, width and height of the prism must be 1, 3, and 7, in some order.
The sum of these is $1 + 3 + 7 = 11$.

Answer: (A)

13. Since $8 = 2 \times 2 \times 2 = 2^3$, then $8^{20} = (2^3)^{20} = 2^{3\times 20} = 2^{60}$.
Thus, if $2^n = 8^{20}$, then $n = 60$.

Answer: (B)

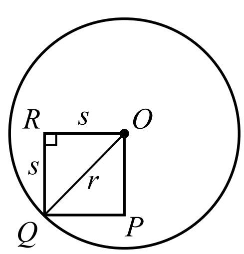
14. Since $3 \times 5 \times 7 = 105$, then the greatest possible value of n is *at least* 105.
In particular, the greatest possible value of n must be positive.
For the product of three numbers to be positive, either all three numbers are positive (that is, none of the numbers is negative) or one number is positive and two numbers are negative. (If there were an odd number of negative factors, the product would be negative.)
If all three numbers are positive, the product is as large as possible when the three numbers are each as large as possible. In this case, the greatest possible value of n is $3 \times 5 \times 7 = 105$.
If one number is positive and two numbers are negative, their product is as large as possible if the positive number is as large as possible (7) and the product of the two negative numbers is as large as possible.
The product of the two negative numbers will be as large as possible when the negative numbers are each "as negative as possible" (that is, as far from 0 as possible). In this case, these numbers are thus -4 and -6 with product $(-4)\times(-6) = 24$. (We can check the other possible products of two negative numbers and see that none is as large.)
So the greatest possible value of n in this case is $7 \times (-4) \times (-6) = 7 \times 24 = 168$.
Combining the two cases, we see that the greatest possible value of n is 168.

Answer: (A)

15. Since the ratio of green marbles to yellow marbles to red marbles is $3 : 4 : 2$, then we can let the numbers of green, yellow and red marbles be $3n$, $4n$ and $2n$ for some positive integer n.
Since 63 of the marbles in the bag are not red, then the sum of the number of green marbles and the number of yellow marbles in the bag is 63.
Thus, $3n + 4n = 63$ and so $7n = 63$ or $n = 9$, which means that the number of red marbles in the bag is $2n = 2 \times 9 = 18$.

Answer: (B)

16. Let s be the side length of the square. Therefore, $OR = RQ = s$.
Let r be the radius of the circle. Therefore, $OQ = r$ since O is the centre of the circle and Q is on the circumference of the circle.
Since the square has a right-angle at each of its vertices, then $\triangle ORQ$ is right-angled at R.



By the Pythagorean Theorem, $OR^2 + RQ^2 = OQ^2$ and so $s^2 + s^2 = r^2$ or $2s^2 = r^2$.
In terms of r, the area of the circle is πr^2.
Since we are given that the area of the circle is 72π, then $\pi r^2 = 72\pi$ or $r^2 = 72$.
Since $2s^2 = r^2 = 72$, then $s^2 = 36$.
In terms of s, the area of the square is s^2, so the area of the square is 36.

Answer: (E)

17. Suppose that Carley buys x boxes of chocolates, y boxes of mints, and z boxes of caramels.
In total, Carley will then have $50x$ chocolates (since there are 50 chocolates in a box of chocolates), $40y$ mints (since there are 40 mints in a box of mints), and $25z$ caramels (since there are 25 caramels in a box of caramels).
Since the contents of each bag was the same and Carley made no incomplete treat bags and there were no left-over candies, then it must be the case that $50x = 40y = 25z$.
We want to find the minimum possible positive value of $x + y + z$ given this condition.
Dividing by the common factor of 5, the equation $50x = 40y = 25z$ becomes $10x = 8y = 5z$.
Since $10x$ is a multiple of 10 and $8y$ is a multiple 8 and $10x = 8y$, we look for the smallest multiple of 10 which is also a multiple of 8.
Since 10, 20 and 30 are not multiples of 8, and 40 is a multiple of 8, then the smallest possible value of $10x$ appears to be 40.
In this case, $x = 4$, $y = 5$ and $z = 8$ gives $10x = 8y = 5z = 40$, and these are the smallest positive integers that create this equality.
Since x, y and z are each the smallest possible, then their sum $x + y + z$ is also the smallest possible.
Thus, the minimum number of boxes that Carley could have bought is $4 + 5 + 8 = 17$.

Answer: (B)

18. *Solution 1*
Suppose that when Nate arrives on time, his drive takes t hours.
When Nate arrives 1 hour early, he arrives in $t - 1$ hours.
When Nate arrives 1 hour late, he arrives in $t + 1$ hours.
Since the distance that he drives is the same in either case and distance equals speed multiplied by time, then $(60 \text{ km/h}) \times ((t - 1) \text{ h}) = (40 \text{ km/h}) \times ((t + 1) \text{ h})$.
Expanding, we obtain $60t - 60 = 40t + 40$ and so $20t = 100$ or $t = 5$.
The total distance that Nate drives is thus $(60 \text{ km/h}) \times (4 \text{ h}) = 240$ km.
When Nate drives this distance in 5 hours at a constant speed, he should drive at $\dfrac{240 \text{ km}}{5 \text{ h}}$ which equals 48 km/h.

Solution 2
Suppose that the distance that Nate drives is d km.
Since driving at 40 km/h causes Nate to arrive 1 hour late and driving at 60 km/h causes Nate to arrive 1 hour early, then the difference between the lengths of time at these two speeds is 2 hours.
Since time equals distance divided by speed, then

$$\frac{d \text{ km}}{40 \text{ km/h}} - \frac{d \text{ km}}{60 \text{ km/h}} = 2 \text{ h}$$

Multiplying both sides of the equation by 120 km/h, we obtain

$$3d \text{ km} - 2d \text{ km} = 240 \text{ km}$$

which gives us $d = 240$.
Thus, the distance that Nate drives is 240 km.
At 40 km/h, the trip takes 6 hours and Nate arrives 1 hour late.
To arrive just in time, it should take 5 hours.
To drive 240 km in 5 hours, Nate should drive at a constant speed of $\dfrac{240 \text{ km}}{5 \text{ h}} = 48$ km/h.

ANSWER: (D)

19. For each of the 10 questions, each correct answer is worth 5 points, each unanswered question is worth 1 point, and each incorrect answer is worth 0 points.
If 10 of 10 questions are answered correctly, the total score is $10 \times 5 = 50$ points.
If 9 of 10 questions are answered correctly, either 0 or 1 questions can be unanswered. This means that the total score is either $9 \times 5 = 45$ points or $9 \times 5 + 1 = 46$ points.
If 8 of 10 questions are answered correctly, either 0 or 1 or 2 questions can be unanswered. This means that the total score is either $8 \times 5 = 40$ points or $8 \times 5 + 1 = 41$ points or $8 \times 5 + 2 = 42$ points.
If 7 of 10 questions are answered correctly, either 0 or 1 or 2 or 3 questions can be unanswered. This means that the total score is one of 35, 36, 37, or 38 points.
If 6 of 10 questions are answered correctly, either 0 or 1 or 2 or 3 or 4 questions can be unanswered. This means that the total score is one of 30, 31, 32, 33, or 34 points.
So far, we have seen that the following point totals between 30 and 50, inclusive, are possible:

$$30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 45, 46, 50$$

which means that

$$39, 43, 44, 47, 48, 49$$

are not possible.
If 5 or fewer questions are answered correctly, is it possible to obtain a total of at least 39 points?
The answer is no, because in this case, the number of correct answers is at most 5 and the number of unanswered questions is at most 10 (these both can't happen at the same time) which together would give at most $5 \times 5 + 10 = 35$ points.
Therefore, there are exactly 6 integers between 30 and 50, inclusive, that are not possible total scores.

ANSWER: (D)

20. We determine when $3^m + 7^n$ is divisible by 10 by looking at the units (ones) digits of $3^m + 7^n$. To do this, we first look individually at the units digits of 3^m and 7^n.

The units digits of powers of 3 cycle $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
To see this, we note that the first few powers of 3 are

$$3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243 \quad 3^6 = 729$$

Since the units digit of a product of integers depends only on the units digits of the integers being multiplied and we multiply by 3 to get from one power to the next, then once a units digit recurs in the sequence of units digits, the following units digits will follow the same pattern.
This means that the units digits of powers of 3 cycle every four powers of 3.
Therefore, of the 100 powers of 3 of the form 3^m with $1 \leq m \leq 100$, exactly 25 will have a units digit of 3, exactly 25 will have a units digit of 9, exactly 25 will have a units digit of 7, and exactly 25 will have a units digit of 1.

The units digits of powers of 7 cycle $7, 9, 3, 1, 7, 9, 3, 1, \ldots$.
To see this, we note that the first few powers of 7 are

$$7^1 = 7 \quad 7^2 = 49 \quad 7^3 = 343 \quad 7^4 = 2401 \quad 7^5 = 16\,807 \quad 7^6 = 117\,649$$

Using the same argument as above, the units digits of powers of 7 cycle every four powers of 7.
Since 101 is 1 more than a multiple of 4, then the power 7^{101} is at the beginning of one of these cycles, and so the units digit of 7^{101} is a 7.
Therefore, of the 105 powers of 7 of the form 7^n with $101 \leq n \leq 205$, exactly 27 will have a units digit of 7, exactly 26 will have a units digit of 9, exactly 26 will have a units digit of 3, and exactly 26 will have a units digit of 1. (Here, 105 powers include 26 complete cycles of 4 plus one additional term.)

For $3^m + 7^n$ to have a units digit of 0 (and thus be divisible by 10), one of the following must be true:

- the units digit of 3^m is 3 (25 possible values of m) and the units digit of 7^n is 7 (27 possible values of n), or
- the units digit of 3^m is 9 (25 possible values of m) and the units digit of 7^n is 1 (26 possible values of n), or
- the units digit of 3^m is 7 (25 possible values of m) and the units digit of 7^n is 3 (26 possible values of n), or
- the units digit of 3^m is 1 (25 possible values of m) and the units digit of 7^n is 9 (26 possible values of n).

The number of possible pairs (m, n) is therefore

$$27 \times 25 + 26 \times 25 + 26 \times 25 + 26 \times 25 = 25 \times (27 + 26 + 26 + 25) = 25 \times 105 = 2625$$

ANSWER: (E)

21. To determine the number of points (x, y) on the line with equation $y = 4x + 3$ that lie inside this region, we determine the number of integers x with $25 \leq x \leq 75$ that have the property that $y = 4x + 3$ is an integer between 120 and 250.
In other words, we determine the number of integers x with $25 \leq x \leq 75$ for which $120 \leq 4x + 3 \leq 250$ and is an integer.
We note that, as x increases, the value of the expression $4x + 3$ increases.
Also, when $x = 29$, we get $4x + 3 = 119$, and when $x = 30$, we get $4x + 3 = 123$.
Further, when $x = 61$, we get $4x + 3 = 247$, and when $x = 62$, we get $4x + 3 = 251$.
Therefore, $4x + 3$ is between 120 and 250 exactly when $30 \leq x \leq 61$.
There are $61 - 30 + 1 = 32$ such values of x, and so there are 32 points that satisfy the given conditions.

Answer: (D)

22. Since $PT = 1$ and $TQ = 4$, then $PQ = PT + TQ = 1 + 4 = 5$.
$\triangle PSQ$ is right-angled at S and has hypotenuse PQ.
We can thus apply the Pythagorean Theorem to obtain $PS^2 = PQ^2 - QS^2 = 5^2 - 3^2 = 16$.
Since $PS > 0$, then $PS = 4$.
Consider $\triangle PSQ$ and $\triangle RTQ$.
Each is right-angled and they share a common angle at Q. Thus, these two triangles are similar.
This tells us that $\dfrac{PQ}{QS} = \dfrac{QR}{TQ}$.
Using the lengths that we know, $\dfrac{5}{3} = \dfrac{QR}{4}$ and so $QR = \dfrac{4 \cdot 5}{3} = \dfrac{20}{3}$.
Finally, $SR = QR - QS = \dfrac{20}{3} - 3 = \dfrac{11}{3}$.

Answer: (B)

23. Suppose that N is an integer that satisfies the given properties.
The first digit of N must be a 1, since there must be at least one 1 before the first 2, at least one 2 before the first 3, and at least one 3 before the 4, which means that we cannot have a 2, a 3, or a 4 before the first 1.
Since there are three 1s in N, then N can begin with 1, 11 or 111.
The first digit of N that is not a 1 must be a 2.
Thus, N begins 12, 112, or 1112.

Case 1: N begins 12
Since no two 2s can be next to each other, we next place the remaining two 2s.
Reading from the left, these 2s can go in positions 4 and 6, 4 and 7, 4 and 8, 4 and 9, 5 and 7, 5 and 8, 5 and 9, 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 10 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 5 positions left open.
Next, we place the remaining two 1s.
If we call these 5 positions A, B, C, D, E, we see that there are 10 pairs of positions in which the 1s can be placed: A and B, A and C, A and D, A and E, B and C, B and D, B and E, C and D, C and E, D and E.
This leaves 3 positions in which the two 3s and one 4 must be placed. Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders (3 and 4, or 4 and 3).
In this case, there are $10 \times 10 \times 2 = 200$ possible integers N.

Case 2: N begins 112
Since no two 2s can be next to each other, we next place the remaining 2s.
Reading from the left, these 2s can go in positions 5 and 7, 5 and 8, 5 and 9, 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 6 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 4 positions left open.
Next, we place the remaining 1. There are 4 possible positions in which the 1 can be placed.
This leaves 3 positions in which the two 3s and one 4 must be placed. Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders.
In this case, there are $6 \times 4 \times 2 = 48$ possible integers N.

Case 3: N begins 1112
Since no two 2s can be next to each other, we next place the remaining 2s.
Reading from the left, these 2s can go in positions 6 and 8, 6 and 9, or 7 and 9.
In other words, there are 3 possible pairs of positions in which the 2s can be placed.
Once the 2s are placed, there are 3 positions left open with only the two 3s and the 4 remaining to be placed.
Reading from the left, a 3 must be placed in the first empty position, since there must be a 3 before the 4.
This leaves 2 positions, in which the remaining digits (3 and 4) can be placed in any order; there are 2 such orders.
In this case, there are $3 \times 2 = 6$ possible integers N.

Combining the three cases, we see that there are $200 + 48 + 6 = 254$ possible integers N.

Answer: (C)

24. Suppose that $GP = x$.
Since the edge length of the cube is 200, then $HP = 200 - x$.
We consider tetrahedron (that is, triangle-based pyramid) $FGMP$ and calculate its volume in two different ways.
The volume of a tetrahedron is equal to one-third times the area of its triangular base times the length of its perpendicular height.
First, we consider tetrahedron $FGMP$ as having base $\triangle FGM$ and height GP, which is perpendicuar to the base.
$\triangle FGM$ is right-angled at G and has $FG = GM = 200$, so its area is $\frac{1}{2} \times FG \times GM$ which equals $\frac{1}{2} \times 200 \times 200$ which equals 20 000.
Thus, the volume of $FGMP$ is $\frac{1}{3} \times 20\,000 \times x$.

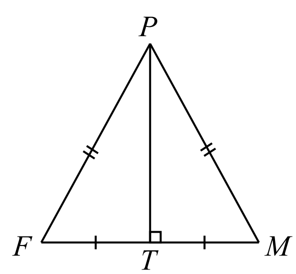
Next, we consider tetrahedron $FGMP$ as having base $\triangle PFM$.
From the given information, the shortest distance from G to a point inside this triangle is 100.
This means that the height of tetrahedron $FGMP$ when considered to have base $\triangle PFM$ is 100.
We need to calcualte the area of $\triangle PFM$.
Since $\triangle FGM$ is right-angled at G and $FM > 0$, then by the Pythagorean Theorem,

$$FM = \sqrt{FG^2 + GM^2} = \sqrt{200^2 + 200^2} = \sqrt{200^2 \times 2} = 200\sqrt{2}$$

Since $\triangle FGP$ is right-angled at G and $FP > 0$, then by the Pythagorean Theorem,

$$FP = \sqrt{FG^2 + GP^2} = \sqrt{200^2 + x^2} = \sqrt{x^2 + 40\,000}$$

Similarly, $MP = \sqrt{x^2 + 40\,000}$.
This means that $\triangle PFM$ is isosceles with $FP = MP$.
Let T be the midpoint of FM.
Then $FT = TM = 100\sqrt{2}$.
Since $\triangle PFM$ is isosceles, then PT is perpendicular to FM.



By the Pythagorean Theorem,

$$PT = \sqrt{FP^2 - FT^2} = \sqrt{\left(\sqrt{x^2 + 40\,000}\right)^2 - \left(100\sqrt{2}\right)^2} = \sqrt{x^2 + 40\,000 - 20\,000} = \sqrt{x^2 + 20\,000}$$

Therefore, the area of $\triangle PFM$ is $\frac{1}{2} \times FM \times PT$ which equals $\frac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}$.
This means that the volume of tetrahedron $FGMP$ is equal to

$$\frac{1}{3} \times \left(\frac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100$$

We can now equate the two expressions for the volume of $FGMP$ to solve for x:

$$\begin{aligned}
\tfrac{1}{3} \times 20\,000 \times x &= \tfrac{1}{3} \times \left(\tfrac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100 \\
20\,000 \times x &= \left(\tfrac{1}{2} \times 200\sqrt{2} \times \sqrt{x^2 + 20\,000}\right) \times 100 \\
x &= \tfrac{1}{2} \times \sqrt{2} \times \sqrt{x^2 + 20\,000} \\
2x &= \sqrt{2} \times \sqrt{x^2 + 20\,000} \\
4x^2 &= 2(x^2 + 20\,000) \\
2x^2 &= 40\,000 \\
x^2 &= 20\,000
\end{aligned}$$

Since $x > 0$, then $x = \sqrt{20\,000} = \sqrt{10\,000 \times 2} = \sqrt{100^2 \times 2} = 100\sqrt{2}$.
This means that $HP = 200 - x = 200 - 100\sqrt{2} \approx 58.58$.
Of the given answers, this is closest to 59, which is (D).

Answer: (D)

25. We will use the result that if a positive integer N has prime factorization $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for some distinct prime numbers $p_1, p_2, \ldots, p_k$ and positive integers $a_1, a_2, \ldots, a_k$, then N has exactly $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors.

This result is based on the following facts:

F1. Every positive integer greater than 1 can be written as a product of prime numbers in a unique way. (If the positive integer is itself prime, this product consists of only the prime number.) This fact is called the "Fundamental Theorem of Arithmetic". This fact is often seen implicitly in finding a "factor tree" for a given integer. For example, 1500 is equal to $2^2 \times 3^1 \times 5^3$ and there is no other way of writing 1500 as a product of prime numbers. Note that rearranging the same prime factors in a different order does not count as a different factorization.

F2. If n is a positive integer and d is a positive integer that is a divisor of n, then the only possible prime factors of d are the prime factors of n. For example, if d is a positive divisor of $n = 1500$, then the only possible prime factors of d are 2, 3 and 5. This means, for example, that d cannot be divisible by 7 or by 11 or by any other prime number not equal to 2, 3 or 5. d might or might not be divisible by each of 2, 3 or 5.

F3. If n is a positive integer, d is a positive integer that is a divisor of n, and p is a prime factor of both n and d, then p cannot divide d "more times" than it divides n. For example, if d is a positive divisor of $n = 1500 = 2^2 \times 3^1 \times 5^3$ that is divisible by 5, then d can be divisible by 5 or by 5^2 or by 5^3 but cannot be divisible by 5^4 or by 5^5 or by any larger power of 5.

From these facts, the positive divisors of $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ are the integers of the form

$$d = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

where $b_1, b_2, \ldots, b_k$ are non-negative integers with $0 \le b_1 \le a_1$, $0 \le b_2 \le a_2$, and so on.
This means that there are $a_1 + 1$ possible values for b_1, namely $0, 1, 2, \ldots, a_1$.
Similarly, there are $a_2 + 1$ possible values for b_2, $a_3 + 1$ possible values for b_3, and so on.
Since every combination of these possible values gives a different divisor d, then there are $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors.

Suppose that n has prime factorization $2^r 5^s p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$ for some distinct prime numbers $p_3, p_4, \ldots, p_k$ none of which equal 2 or 5, for some positive integers $a_3, a_4, \ldots, a_k$, and some non-negative integers r and s.
We have written n this way to allow us to pay special attention to the possible prime factors of 2 and 5.
This means that

$$2n = 2^{r+1}5^s p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$$
$$5n = 2^r 5^{s+1} p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$$

Since $2n$ has 64 positive divisors and $5n$ has 60 positive divisors, then

$$(r+2)(s+1)(a_3+1)(a_4+1)\cdots(a_k+1) = 64$$
$$(r+1)(s+2)(a_3+1)(a_4+1)\cdots(a_k+1) = 60$$

Since every factor in the two expressions on the left is a positive integer, then

$$(a_3+1)(a_4+1)\cdots(a_k+1)$$

is a positive common divisor of 64 and of 60.
The positive divisors of 64 are 1, 2, 4, 8, 16, 32, 64.
Of these, only 1, 2, 4 are divisors of 60.
Therefore, $(a_3+1)(a_4+1)\cdots(a_k+1)$ equals 1, 2 or 4.
Since each of $a_3, a_4, \ldots, a_k$ is a positive integer, then each of $a_3+1, a_4+1, \ldots, a_k+1$ is at least 2.

Case 1: $(a_3+1)(a_4+1)\cdots(a_k+1) = 4$
The only ways in which 4 can be written as a product of positive integers each at least 2 are 2×2 and 4 (a product of one integer is itself).
Thus, either $k = 4$ with $a_3 + 1 = a_4 + 1 = 2$ (giving $a_3 = a_4 = 1$), or $k = 3$ with $a_3 + 1 = 4$ (giving $a_3 = 3$).
Since

$$(r+2)(s+1)(a_3+1)(a_4+1)\cdots(a_k+1) = 64$$
$$(r+1)(s+2)(a_3+1)(a_4+1)\cdots(a_k+1) = 60$$

then, after simplification, we have

$$(r+2)(s+1) = 16$$
$$(r+1)(s+2) = 15$$

Expanding the left sides of the two equations, we obtain

$$rs + r + 2s + 2 = 16$$
$$rs + 2r + s + 2 = 15$$

Subtracting the second of these equations from the first, we obtain $-r + s = 1$ and so $s = r + 1$.
Substituting into the equation $(r+2)(s+1) = 16$, we obtain $(r+2)(r+2) = 16$.
Since $r > 0$, then $(r+2)^2 = 16$ gives $r + 2 = 4$ and so $r = 2$, which gives $s = 3$.
Therefore, we could have $r = 2$, $s = 3$.

Combining with the possible values of a_3 and a_4, this means that we could have $n = 2^2 5^3 p_3 p_4$ for some primes p_3 and p_4 not equal to 2 or 5, or $n = 2^2 5^3 p_3^3$ for some prime p_3 not equal to 2 or 5.

We can verify that $2n$ and $5n$ have the correct number of positive divisors in each case.

Case 2: $(a_3 + 1)(a_4 + 1) \cdots (a_k + 1) = 2$

The only way in which 2 can be written as a product of positive integers each at least 2 is 2.
Thus, $k = 3$ with $a_3 + 1 = 2$ (giving $a_3 = 1$).
Since

$$(r + 2)(s + 1)(a_3 + 1)(a_4 + 1) \cdots (a_k + 1) = 64$$
$$(r + 1)(s + 2)(a_3 + 1)(a_4 + 1) \cdots (a_k + 1) = 60$$

then

$$(r + 2)(s + 1) = 32$$
$$(r + 1)(s + 2) = 30$$

We could proceed as in Case 1.
Alternatively, knowing that r and s are non-negative integers, then the possibilities from the first equation are

$r+2$	$s+1$	r	s	$r+1$	$s+2$	$(r+2)(s+1)$
32	1	30	0	31	2	62
16	2	14	1	15	3	45
8	4	6	3	7	5	35
4	8	2	7	3	9	27
2	16	0	15	1	17	17
1	32	-1	31	0	33	0

There are no values of r and s that work in this case.

Case 3: $(a_3 + 1)(a_4 + 1) \cdots (a_k + 1) = 1$

Since each factor on the left side is supposedly at least 2, what can this mean? This actually means that there are no factors on the left side. In other words, $k = 2$ and $n = 2^r 5^s$.
(See if you can follow the argument before Case 1 through to verify that there are no contradictions.)
Here,

$$(r + 2)(s + 1) = 64$$
$$(r + 1)(s + 2) = 60$$

Knowing that r and s are non-negative integers, then the possibilities from the first equation are

$r+2$	$s+1$	r	s	$r+1$	$s+2$	$(r+2)(s+1)$
64	1	62	0	63	2	126
32	2	30	1	31	3	93
16	4	14	3	15	5	45
8	8	6	7	7	9	63
4	16	2	13	3	15	51
2	32	0	31	1	33	33
1	64	-1	63	0	65	0

There are no values of r and s that work in this case.

Therefore, combining the results of the three cases, the positive integer n satisfies the given conditions exactly when

- $n = 2^2 5^3 p_3 p_4 = 500 p_3 p_4$ for some primes p_3 and p_4 not equal to 2 or 5, or
- $n = 2^2 5^3 p_3^3 = 500 p_3^3$ for some prime p_3 not equal to 2 or 5.

Since $n \leq 20\,000$, then either

- $500 p_3 p_4 \leq 20\,000$ which gives $p_3 p_4 \leq 40$, or
- $500 p_3^3 \leq 20\,000$ which gives $p_3^3 \leq 40$.

It remains to determine the number of pairs of primes p_3 and p_4 that are not equal to 2 or 5 with product less than 40, and the number of primes p_3 that are not equal to 2 or 5 whose cube is less than 40.
In the first case, the possibilities are:

$$3 \times 7 = 21 \qquad 3 \times 11 = 33 \qquad 3 \times 13 = 39$$

The order of p_3 and p_4 does not matter as switching the order gives the same value for n. We note as well that we cannot have both p_3 and p_4 at least 7 and have their product at most 40.
In the second case, the only possibility is $p_3^3 = 3^3$.

This means that there are 4 possible values of n that satisfy the given conditions.

Answer: (A)