The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Canadian Team Mathematics Contest

Answer Key for Team Problems

Question	Answer
1	4
2	6
3	$36°$
4	16665
5	10004444
6	6
7	2880
8	64π
9	11
10	$\frac{1}{2}$
11	15
12	48
13	$(0, 9)$
14	5.6 kg
15	16
16	25π
17	20
18	$\frac{3}{16}$
19	256
20	17
21	-4
22	334
23	$20\sqrt{5}$
24	-4
25	510

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Answer Key for Individual Problems

Question	Answer
1	3
2	2
3	1
4	$\frac{7}{12}$
5	9
6	53
7	6
8	2.5 km
9	$(-7, 5)$
10	$(3, 26, 7, 13), (15, 2, 73, 7)$

Answer Key for Relays

Question	Answer
0	$9, 108, 36^\circ$
1	$12, 4, 15$
2	$6, 5, 13$
3	$40, 30, 20$

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April 2020

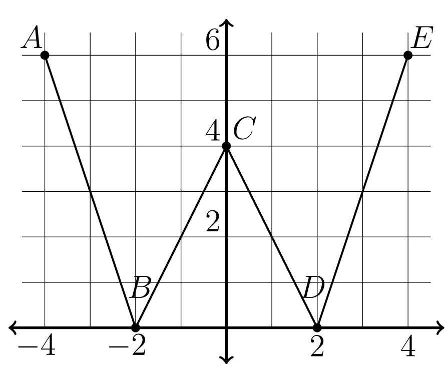
Solutions

Individual Problems

1. *Solution 1*
   The numerator is equal to $24 + 12 = 36$. The denominator is equal to $4^2 - 4 = 16 - 4 = 12$, so
   $$\frac{24+12}{4^2-4} = \frac{36}{12} = 3.$$
   Solution 2
   By factoring 4 out of both the numerator and denominator, we get
   $$\frac{24+12}{4^2-4} = \frac{4(6+3)}{4(4-1)} = \frac{6+3}{4-1} = \frac{9}{3} = 3.$$
   ANSWER: 3

2. *Solution 1*
   By factoring and substituting, we get
   $$\frac{6}{5}k - 2 = \frac{2}{5}(3k) - 2 = \frac{2}{5}(10) - 2 = 4 - 2 = 2.$$
   Solution 2
   Solving for k, we have $k = \frac{10}{3}$, so
   $$\frac{6}{5}k - 2 = \frac{6}{5} \times \frac{10}{3} - 2 = \frac{60}{15} - 2 = 4 - 2 = 2.$$
   ANSWER: 2

3. *Solution 1*
   The slope of segment AB is $\frac{0-6}{-2-(-4)} = -3$.
   Since ED is the reflection of AB in the y-axis, its slope is the negative of the slope of AB. This means the slope of ED is 3.
   Similarly, the slope of segment BC is $\frac{4-0}{0-(-2)} = 2$, so the slope of DC is -2.
   Therefore, the sum of the slopes is $-2 + 3 = 1$.
   Solution 2
   From the information given, E is the reflection in the y-axis of A and D is the reflection in the y-axis of B.
   Since the coordinates of A are $(-4, 6)$, this means the coordinates of E are $(4, 6)$. Similarly, since the coordinates of B are $(-2, 0)$, the coordinates of D are $(2, 0)$.



The slope of segment DC is $\frac{0-4}{2-0} = -2$. The slope of ED is $\frac{6-0}{4-2} = 3$, so the sum of the slopes is $-2+3=1$.

ANSWER: 1

4. *Solution 1*
The odd numbers on the spinner are 1, 3, and 5.
The wedges labelled by 3 and 5 each take up $\frac{1}{4}$ of the spinner and so each will be spun with a probability of $\frac{1}{4}$.
If we let the probability of spinning 1 be x, then we have that the probability of spinning 2 is $2x$.
The probability of spinning either 1 or 2 is $\frac{1}{4}$, which means $x+2x=\frac{1}{4}$ or $3x=\frac{1}{4}$, so $x=\frac{1}{12}$.
Therefore, the probability of spinning an odd number is $\frac{1}{4}+\frac{1}{4}+\frac{1}{12}=\frac{7}{12}$.

Solution 2
The radii defining the sector containing the wedges labelled by 1 and 2 meet at an angle of 90°.
Let y° be the angle made by the radii defining the wedge labelled by 1. Then $y+2y=90$, so $y=30$.
The angles made by the radii defining the wedges labelled by 3 and 5 are both 90°.
Since there are 360° in a circle, this means the probability of spinning an odd number is

$$\frac{30^\circ+90^\circ+90^\circ}{360^\circ}=\frac{210^\circ}{360^\circ}=\frac{7}{12}.$$

ANSWER: $\frac{7}{12}$

5. If two lines are parallel, then they will not intersect unless they are the same line. This means two lines that have the same slope but different y-intercepts cannot intersect.
On the other hand, two lines that have different slopes always intersect at exactly one point.

The six lines that Maggie graphs can be broken into two sets of three distinct parallel lines.
As discussed above, each of the three lines with slope 1 intersects each of the three lines with slope -2 exactly once.
This gives a total of $3\times 3=9$ intersections.

We must check that none of the points have been counted more than once. For this to have happened, we would need to have at least three of the six lines intersecting at the same point.
There are only two possible slopes, so among any three of the lines, at least two must have the same slope.
Therefore, if three of the lines intersect at the same point, then two distinct lines of equal slope intersect.
This would mean that two distinct parallel lines intersect, which cannot happen.

Therefore, the nine points are distinct, so $n=9$.

ANSWER: 9

6. The prime factorization of 60 is $2^2\times 3\times 5$, so the prime factorization of 60^5 is

$$(2^2\times 3\times 5)^5 = 2^{10}3^5 5^5.$$

Therefore, the divisors of 60^5 are numbers of the form $2^a3^b5^c$ where a, b, and c are integers with $0\le a\le 10$, $0\le b\le 5$, and $0\le c\le 5$.
A divisor is a perfect square exactly when each of a, b, and c is even.
Therefore, the divisors of 60^5 that are perfect squares are the numbers of the form $2^a3^b5^c$ where

a is equal to 0, 2, 4, 6, 8, or 10, b is equal to 0, 2, or 4, and c is equal to 0, 2, or 4.
This gives a total of $6 \times 3 \times 3 = 54$ ways of choosing a, b, and c so that $2^a 3^b 5^c$ is a perfect square.
This total includes the divisor obtained by setting $a = b = c = 0$, which is $2^0 3^0 5^0 = 1$.
We want the number of perfect-square divisors greater than 1, so the answer is $54 - 1 = 53$.

ANSWER: 53

7. Since $\sqrt[3]{27} = 3$, the dimensions of the larger cube must be $3 \times 3 \times 3$.
Therefore, each side of the larger cube has area $3 \times 3 = 9$.
A cube has 6 faces, so the total surface of the cube is made up of $9 \times 6 = 54$ of the 1×1 squares from the faces of the unit cubes.
Since $\frac{1}{3}$ of the surface area is red, this means $\frac{54}{3} = 18$ of these unit squares must be red.

The unit cube at the centre of the larger cube has none of its faces showing, the 6 unit cubes in the centres of the outer faces have exactly 1 face showing, the 12 unit cubes on an edge but not at a corner have 2 faces showing, one of each of two adjacent sides, and the 8 unit cubes at the corners each have 3 faces showing.
For any unit cube, there are either 0, 1, 2, or 3 of its faces showing on the surface of the larger cube.
This means at most three faces of any unit cube are on the surface of the larger cube. Thus, there must be at least 6 cubes painted red in order to have 18 red 1×1 squares on the surface of the larger cube.

There are 8 unit cubes on the corners, so if we colour exactly 6 unit cubes red and the other 21 black, then arrange the cubes into a $3 \times 3 \times 3$ cube so that the 6 red unit cubes are at corners, there will be exactly 18 of the 1×1 squares on the surface coloured red.
Therefore, the answer is 6.

ANSWER: 6

8. Let d_1 be the distance in kilometres that Gina had ran when her average rate was 7 minutes 30 seconds per kilometre. Let d_2 be the distance in kilometres that Gina had ran when her average rate was 7 minutes 5 seconds per kilometre.
The distance we seek is $d_2 - d_1$.

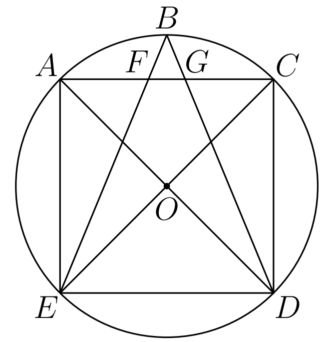
Since 7 minutes 30 seconds was the average rate when the distance was d_1 kilometres, $\frac{15}{2} \times d_1$ minutes passed from the time Gina started the app to when she had ran d_1 kilometres. We are using that 7 minutes 30 seconds is equal to $\frac{15}{2}$ minutes.
On the other hand, Gina ran at a constant rate of 7 minutes per kilometre, so it took $7d_1$ minutes for her to run d_1 kilometres once she started moving.
Since 15 seconds equals $\frac{1}{4}$ minute, this means that $\frac{1}{4} + 7d_1$ minutes passed from the time Gina started the app to the time she had ran d_1 kilometres.
This gives the equation $\frac{15}{2}d_1 = \frac{1}{4} + 7d_1$ which can be solved for d_1 to get $d_1 = \frac{1}{2}$ kilometre.

Since 5 seconds equals $\frac{1}{12}$ minute, 7 minutes 5 seconds equals $\frac{85}{12}$ minutes.
Using calculations similar to those involving d_1, we can obtain the equation $\frac{85}{12}d_2 = \frac{1}{4} + 7d_2$, which can be solved for d_2 to get $d_2 = 3$ kilometres.

Therefore, the answer is $d_2 - d_1 = 3 - \frac{1}{2} = \frac{5}{2}$ kilometres or 2.5 kilometres.

ANSWER: 2.5 km

9. Connect O to each of A, C, D, and E as shown.
Each of OA, OC, OD, and OE are radii of the circle and so are equal.
As well, $AC = CD = DE = EA$ since they are the side-lengths of a square, which means $\triangle AOC$, $\triangle COD$, $\triangle DOE$, and $\triangle EOA$ are all congruent.
This means



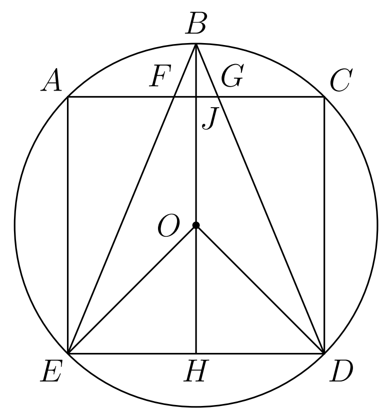
$$\angle AOC = \angle COD = \angle DOE = \angle EOA$$

and since the sum of these angles must be 360°,

$$\angle AOC = \angle COD = \angle DOE = \angle EOA = \frac{360^\circ}{4} = 90^\circ.$$

By the Pythagorean theorem, $OE^2 + OD^2 = ED^2$.
Since the radius of the circle is $\sqrt{2}$, we have $(\sqrt{2})^2 + (\sqrt{2})^2 = ED^2$ or $2 + 2 = 4 = ED^2$, so $ED = 2$ since $ED > 0$.

Now draw a line from B through O and label by H and J its intersections with ED and AC, respectively.
We know that $\angle EOD = 90^\circ$, which means



$$\angle EOB + \angle DOB = 360^\circ - 90^\circ = 270^\circ.$$

Furthermore, OE, OD, and OB are all radii of the circle and hence are equal, and $BE = BD$ by construction, so $\triangle EOB$ and $\triangle DOB$ are congruent.
This means $\angle EOB = \angle DOB$ and since their sum is 270°, we must have $\angle EOB = \frac{270^\circ}{2} = 135^\circ$.
Since BOH is a straight line, this means $\angle EOH = 180^\circ - 135^\circ = 45^\circ$.
We previously established that $\triangle EOD$ is right and isosceles, so $\angle OED = 45^\circ$ as well.
This means $\angle OHE = 180^\circ - 45^\circ - 45^\circ = 90^\circ$.

We now have that $\triangle EBH$ and $\triangle DBH$ are right triangles having a common hypotenuse length and a shared side, which means they are congruent.
Therefore, $EH = HD$ and it follows that $EH = 1$ because $ED = 2$.

Line segments AC and DE are parallel because they are opposite sides of a square.
Therefore, $\angle BFJ = \angle BEH$ and $\angle BJF = \angle BHE$. It follows that $\triangle BFJ$ and $\triangle BEH$ are similar.

Since $\angle BHE = 90^\circ$, we also have that AE and BH are parallel which means $\angle AEF = \angle JBF$ by alternating angles.
Angles $\angle AFE$ and $\angle JFB$ are opposite and hence equal, and both $\angle EAF$ and $\angle BJF$ are right, which means $\triangle EFA$ and $\triangle BFJ$ are similar.
By similar reasoning, $\triangle DGC$ is also similar to $\triangle BFJ$, $\triangle BEH$, and $\triangle EFA$.
Since $\triangle DGC$ and $\triangle EFA$ are similar and $AE = CD$, we have that $\triangle DGC$ and $\triangle EFA$ are congruent, which means $AF = GC$.

Since $\triangle EFA$ is similar to $\triangle BEH$, we get that

$$\frac{AF}{AE} = \frac{HE}{HB} = \frac{HE}{HO + OB} = \frac{1}{1 + \sqrt{2}}.$$

Since $AE = 2$, this means

$$AF = \frac{2}{1+\sqrt{2}} = \frac{2}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = 2\sqrt{2} - 2.$$

Since $AF = GC$, we have that $GC = 2\sqrt{2} - 2$ as well, and using that $AC = 2$, we have

$$FG = AC - AF - GC = 2 - (2\sqrt{2} - 2) - (2\sqrt{2} - 2) = 6 - 4\sqrt{2}.$$

Since AC and ED are parallel, we have that $\triangle BFG$ is similar to $\triangle BED$.
This means their heights have the same ratio as their sides, so

$$\frac{BJ}{BH} = \frac{FG}{ED}.$$

Since $BH = 1 + \sqrt{2}$, $FG = 6 - 4\sqrt{2}$, and $ED = 2$, this means

$$BJ = BH\frac{FG}{ED} = (1+\sqrt{2})\frac{6-4\sqrt{2}}{2} = (1+\sqrt{2})(3-2\sqrt{2}) = -1+\sqrt{2}.$$

Therefore, the area of $\triangle BFG$ is

$$\frac{1}{2} \times FG \times BJ = \frac{1}{2}(6-4\sqrt{2})(-1+\sqrt{2}) = (3-2\sqrt{2})(-1+\sqrt{2}) = -7+5\sqrt{2}.$$

The answer is $(-7, 5)$.

Answer: $(-7, 5)$

10. Consider the first equation, $ab + 2a - b = 58$.
Factoring a out of the first two terms, we get $a(b+2) - b = 58$.
If the left side of this equation were instead $a(b+2) - b - 2$, we could further factor the left side of the equation to $(a-1)(b+2)$.
To achieve this, we add -2 to both sides of the first equation to get

$$(a-1)(b+2) = 58 - 2 = 56 = 2^3 \times 7.$$

In a similar way, we can add 8 and -24 to both sides of the second and third equations, respectively and factor to get

$$(a-1)(b+2) = 56 \quad (1)$$
$$(b+2)(c+4) = 308 \quad (2)$$
$$(c+4)(d-6) = 77 \quad (3)$$

We are given that c and d are positive integers, which means $c + 4$ is a positive integer and $d - 6$ is an integer.
From (3) and since c is positive, we can see that $c + 4$ is a positive divisor of $77 = 7 \times 11$.
This means that $c + 4$ is one of 1, 7, 11, and 77.

We cannot have $c + 4 = 1$ since this would mean $c = -3$, which is negative.

If $c + 4 = 7$, then $c = 7 - 4 = 3$ and rearranging (2) gives

$$b + 2 = \frac{308}{c+4} = \frac{308}{7} = 44.$$

Rearranging (1), this leads to

$$a - 1 = \frac{56}{b+2} = \frac{56}{44} = \frac{14}{11}$$

which is not an integer, but $a - 1$ must be an integer because a is an integer.
Therefore, we cannot have $c + 4 = 7$.

If $c + 4 = 11$, then (3) implies

$$d - 6 = \frac{77}{c+4} = \frac{77}{11} = 7$$

which means $d = 7 + 6 = 13$.
From (2),

$$b + 2 = \frac{308}{c+4} = \frac{308}{11} = 28,$$

so $b = 28 - 2 = 26$.
Finally, using (1), we get

$$a - 1 = \frac{56}{b+2} = \frac{56}{28} = 2,$$

so $a = 1 + 2 = 3$.
Therefore, $(a, b, c, d) = (3, 26, 7, 13)$ is one possible solution.

If $c+4 = 77$, then similar calculations to those above lead to $d - 6 = \frac{77}{77} = 1$, $b + 2 = \frac{308}{77} = 4$, and $a - 1 = \frac{56}{4} = 14$.
Therefore, $c = 77 - 4 = 73$, $d = 1 + 6 = 7$, $b = 4 - 2 = 2$, and $a = 14 + 1 = 15$.
Thus, the only other solution is $(a, b, c, d) = (15, 2, 73, 7)$.

Answer: $(3, 26, 7, 13)$, $(15, 2, 73, 7)$

Team Problems

1. Since $3^2 = 9 < 11 < 16 = 4^2$, it must be that $3 < \sqrt{11} < 4$.
   Similarly, $4^2 = 16 < 19 < 25 = 5^2$, so $4 < \sqrt{19} < 5$.
   Therefore, $\sqrt{11} < 4 < \sqrt{19}$.

   Answer: 4

2. By factoring 3^3 out of the numerator, we have

$$\frac{3^5 - 3^4}{3^3} = \frac{3^3(3^2 - 3)}{3^3} = 3^2 - 3 = 9 - 3 = 6.$$

   Answer: 6

3. Let the measure of the smallest interior angle be x°.
   Then the other three interior angles have measures $2x^\circ$, $3x^\circ$, and $4x^\circ$.
   The interior angles of a quadrilateral add to 360°, so $360^\circ = x^\circ + 2x^\circ + 3x^\circ + 4x^\circ = 10x^\circ$.
   Therefore, $360 = 10x$, so $x = 36$.

   Answer: 36°

4. *Solution 1*
   Let S be the sum in the problem. We can group terms and rewrite the sum as

$$\begin{aligned} S &= (1010 + 2020) + (1111 + 1919) + (1212 + 1818) + (1313 + 1717) + (1414 + 1616) + 1515 \\ &= 3030 + 3030 + 3030 + 3030 + 3030 + 1515 \\ &= 5 \times 3030 + 1515 \\ &= 16\,665. \end{aligned}$$

   Solution 2
   The numbers being added are the multiples of 101 from 10×101 through 20×101.
   Thus, if we set S to be the sum in the problem, we can factor out 101 to get

$$S = 101(10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20).$$

   It can be checked by hand that $10 + 11 + \cdots + 19 + 20 = 165$. Alternatively, using the fact that

$$1 + 2 + 3 + 4 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

   for any positive integer n, the sum can be computed as

$$\begin{aligned} 10 + 11 + \cdots + 19 + 20 &= (1 + 2 + 3 + \cdots + 19 + 20) - (1 + 2 + \cdots + 8 + 9) \\ &= \frac{20(21)}{2} - \frac{9(10)}{2} \\ &= 10 \times 21 - 9 \times 5 \\ &= 210 - 45 \\ &= 165. \end{aligned}$$

   Therefore, $S = 101 \times 165 = (100 + 1) \times 165 = 16\,500 + 165 = 16\,665$.

   Answer: 16 665

5. The smallest eight-digit number is 10 000 000.
The smallest 10 000 eight-digit numbers are those of the form $10\,00a\,bcd$ where a, b, c, and d are digits. Among the 10 000 smallest eight-digit numbers, 10 004 444 is the only one that has four digits that are equal to 4.
This means the smallest eight-digit positive integer that has exactly four digits which are 4 is 10 004 444.

ANSWER: 10 004 444

6. There were 200 visitors on Saturday, so there were 100 visitors the day before. Since tickets cost \$9 on Fridays, the total money collected on Friday was \$900.
Therefore, the amount of money collected from ticket sales on the Saturday was $\$900 \times \frac{4}{3} = \$1200$.
Since there were 200 visitors on Saturday, the price of tickets on that particular Saturday was $\frac{\$1200}{200} = \$6$.
In fact, if all conditions are kept the same in this problem but the number of ticket sales on Saturday changes, the answer will still be 6. Can you see why this is true?

ANSWER: 6

7. *Solution 1*
Suppose the number is $abcd$ where a, b, c, and d are digits.
Since the number is divisible by 5, we must have that $d = 0$ or $d = 5$.
The digits are all even, which means $d \neq 5$, so $d = 0$.

The smallest that a can be is 2 since it must be even and greater than 0 (a four-digit number cannot have $a = 0$). So we will try to find such a number with $a = 2$.
In order to be divisible by 9, we must have $a + b + c + d$ divisible by 9. Substituting $a = 2$ and $d = 0$, this means $2 + b + c + 0 = 2 + b + c$ is divisible by 9.
Since b and c are even, $2 + b + c$ is even, which means it cannot equal 9. Thus, we will try to find b and c so that $2 + b + c = 18$, which is the smallest multiple of 9 that is greater than 9.
This equation rearranges to $b + c = 16$. Since b and c are even and satisfy $0 \leq b \leq 9$ and $0 \leq c \leq 9$, the only possibility is $b = c = 8$.

Solution 2
An integer is divisible by both 5 and 9 exactly when it is divisible by 45.
Since we seek a number having only even digits, its last digit is one of $0, 2, 4, 6$, or 8, so the number itself is even.
Thus, we are looking for an even multiple of 45.
An even number is a multiple of 45 exactly when it is a multiple of 90.
This means we are looking for the smallest 4-digit multiple of 90 that has only even digits.
The smallest 4-digit multiple of 90 is 1080, but the first digit of this number is 1, which is odd.
Each of the next 10 multiples of 90 has a first digit equal to 1, so the number we seek must be larger than 2000.
The four-digit multiples of 90 that have a thousands digit of 2 are

$$2070, 2160, 2250, 2340, 2430, 2520, 2610, 2700, 2790, 2880, 2970$$

and the only number in this list that has all even digits is 2880.
Therefore, 2880 is the smallest 4-digit number that is a multiple of 5, a multiple of 9, and has only even digits.

ANSWER: 2880

8. The length of a semicircular arc of diameter d is half of the circumference of a circle of diameter d. Thus the length of a semicircular arc of diameter d is $\frac{\pi d}{2}$.
Since the larger semicircle has diameter 64 units, this means the length of the large semicircular arc is $\frac{\pi d}{2} = \frac{64\pi}{2} = 32\pi$.
The diameter of each small semicircle is $\frac{64}{4} = 16$ units, so each small semicircular arc has length $\frac{16\pi}{2} = 8\pi$.
There are four of these small arcs, so the perimeter of the figure is $32\pi + 4(8\pi) = 64\pi$.
ANSWER: 64π

9. By factoring and using exponent rules, we have $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$.
Therefore, $20^{10} = 1024 \times 10^{10}$, which is the integer 1024 followed by ten zeros.
Thus, 20^{10} has eleven digits that are 0. That is, 10 zeros at the end and one coming from the 1024 at the beginning.
ANSWER: 11

10. Using exponent laws, we have

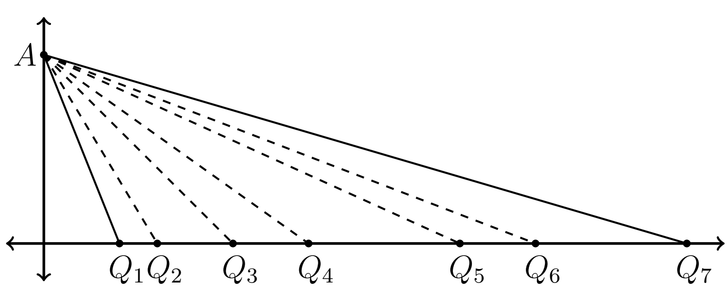
$$\begin{aligned}\frac{16^{x+1}}{8^{2y-1}} &= \frac{(2^4)^{x+1}}{(2^3)^{2y-1}} \\ &= \frac{2^{4(x+1)}}{2^{3(2y-1)}} \\ &= 2^{(4x+4)-(6y-3)} \\ &= 2^{4x-6y+7}\end{aligned}$$

We were given that $3y - 2x = 4$. Multiplying both sides of this equation by -2 leads to $4x - 6y = -8$. Adding 7 to both sides of this equation gives $4x - 6y + 7 = -8 + 7 = -1$.
From above, we have

$$\frac{16^{x+1}}{8^{2y-1}} = 2^{4x-6y+7} = 2^{-1} = \frac{1}{2}.$$

ANSWER: $\frac{1}{2}$

11. Consider the diagram below:



The sum of the areas of the triangles is equal to the area of the larger $\triangle AQ_1Q_7$.
This is because the triangles share vertex A, the base of $\triangle AQ_2Q_3$ begins where the base of $\triangle AQ_1Q_2$ ends, the base of $\triangle AQ_3Q_4$ begins where the base of $\triangle AQ_2Q_3$ ends, and so on, with all of Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, and Q_7 lying on the x-axis.
All of Q_1 through Q_7 lie on the x-axis, so the height of $\triangle AQ_1Q_7$ is the y-coordinate of A, or 2.
The first seven prime numbers are 2, 3, 5, 7, 11, 13, and 17, so $p_1 = 2$ and $p_7 = 17$. The base of $\triangle AQ_1Q_7$ has length $p_7 - p_1 = 17 - 2 = 15$. Therefore, the sum of the areas of the triangles is $\frac{1}{2} \times 15 \times 2 = 15$.
ANSWER: 15

12. Since 90 is a multiple of 3, there are $\frac{90}{3} = 30$ multiples of 3 in the list.
Since 90 is a multiple of 5, there are $\frac{90}{5} = 18$ multiples of 5 in the list.
A multiple of 5 is a multiple of 3 exactly when it is a multiple of 15, which means the multiples of 15 were already crossed out when the multiples of 3 were crossed out.
There are $\frac{90}{15} = 6$ multiples of 15 in the list, so this means there are $18 - 6 = 12$ multiples of 5 remaining in the list once the multiples of 3 have been crossed out.
Therefore, there are $90 - 30 - 12 = 48$ numbers in the list once all of the multiples of 3 and 5 have been crossed out.

Answer: 48

13. By completing the square, we have $y = -\frac{1}{4}(x-10)^2+4$, which means the vertex of the parabola is at $A(10, 4)$.
If we set $y = 0$, we have $-\frac{1}{4}(x-10)^2 + 4 = 0$ or $-\frac{1}{4}(x-10)^2 = -4$, so $(x-10)^2 = 16$.
Taking square roots of both sides, we have $x - 10 = \pm 4$ or $x = 10 \pm 4$.
Therefore, the parabola passes through $(6, 0)$ and $(14, 0)$.
From the information given, this means the coordinates of B are $(6, 0)$ and the coordinates of F are $(14, 0)$.
The second parabola has its vertex at $B(6, 0)$, so its equation is $y = a(x-6)^2$ for some real number a.
Using the fact that it passes through $A(10, 4)$, we get $4 = a(10-6)^2 = 16a$, or $a = \frac{1}{4}$.
Therefore, the equation of the second parabola is $y = \frac{1}{4}(x-6)^2$.
Substituting $x = 0$, we get $y = \frac{1}{4}(-6)^2 = \frac{36}{4} = 9$.
The second parabola crosses the y-axis at $D(0, 9)$.

Answer: $(0, 9)$

14. Since the total mass of the first three fish is 1.5 kg, the average mass of the first three fish is 0.5 kg.
Let M be the total mass of all of the fish. Since the average mass of the first three fish is the same as the average mass of all of the fish, this means $\frac{M}{21} = 0.5$ kg or $M = 10.5$ kg.
Since the first three fish have a total mass of 1.5 kg, this means the last 18 fish that Jeff caught have a total mass of 10.5 kg − 1.5 kg = 9 kg.
If 17 of these 18 fish have as small a mass that is as possible, the 18^{th} of these fish will have a mass that is as large as possible.
The smallest possible mass is 0.2 kg, so the total mass of 17 fish, each having as small a mass as possible, is 17×0.2 kg = 3.4 kg.
The largest possible mass of any fish that Jeff could have caught is 9 kg − 3.4 kg = 5.6 kg.

Answer: 5.6 kg

15. Since $8^2 = 64 < 70$ and $9^2 = 81 > 70$, the largest that any of the integers can possibly be is 8.
The first 8 positive perfect squares are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, and $8^2 = 64$.
The table below records the sum of the squares of any two integers between 1 and 8 inclusive. That is, for example, the cell corresponding to the row for 9 and the column for 25 contains the number $5^2 + 3^2 = 25 + 9 = 34$.

	1	4	9	16	25	36	49	64
1	2	5	10	17	26	37	50	65
4		8	13	20	29	40	53	68
9			18	25	34	45	58	73
16				32	41	52	65	80
25					50	61	74	89
36						72	85	100
49							98	113
64								128

We now note that the sum of the squares of any two of a, b, c, and d must be at least 2, so the sum of any two of a, b, c, and d cannot by more than $70 - 2 = 68$. Therefore, the possible sums of squares of two of a, b, c, and d are the sums from the above table that do not exceed 68:

$$2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 50, 52, 53, 58, 61, 65, 68.$$

To find quadruples (a, b, c, d) satisfying $a^2 + b^2 + c^2 + d^2 = 70$, we can find pairs of numbers in the list above that sum to 70 and then use the table to find a, b c, and d.
The pairs that sum to 70 are 2 and 68, 5 and 65, 17 and 53, 18 and 52, 20 and 50, 25 and 45, 29 and 41.

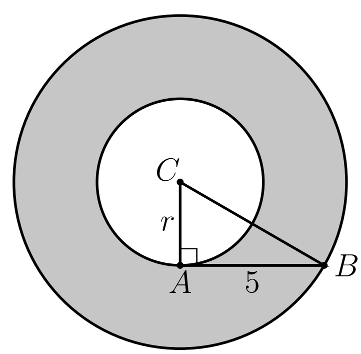
The numbers 2 and 68 come from $1^2 + 1^2$ and $2^2 + 8^2$, so we could have that the four integers a, b, c, and d are 1, 1, 2, and 8.
The numbers 5 and 65 come from $1^2 + 2^2$ and either $1^2 + 8^2$ or $4^2 + 7^2$. This means a, b, c, and d could be either 1, 1, 2, and 8 or 1, 2, 4, and 7. We have already found the first of these possibilities.
Continuing in this way, we must have that a, b, c, and d are one of

$$1, 1, 2, 8;\ \ 1, 2, 4, 7;\ \ 3, 3, 4, 6;\ \ 2, 4, 5, 5$$

and the sums of the entries in these lists are 12, 14, 16, and 16, respectively. Therefore, the largest possible sum is $a + b + c + d = 16$ (and it is achievable in two different ways).

ANSWER: 16

16. Let C be the common centre of the two circles, R be the radius of the larger circle, and r be the radius of the smaller circle.
Since AB is tangent to the smaller circle, we have that AC is perpendicular to AB.
This means $\triangle ABC$ is right-angled at A, so by the Pythagorean theorem and since $AB = 5$, we have $AC^2 + AB^2 = BC^2$ or $r^2 + 5^2 = R^2$.



The area of the shaded region is equal to the area of the larger circle minus the area of the smaller circle, or $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.
Rearranging $r^2 + 25 = R^2$, we have $R^2 - r^2 = 25$, so the area of the shaded region is $\pi(R^2 - r^2) = 25\pi$.

ANSWER: 25π

17. A quadratic function of the form $f(x) = ax^2 + bx + c$ with $a > 0$ attains its minimum when $x = -\dfrac{b}{2a}$.
One way to see this is to complete the square to get

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

and note that since $a\left(x + \frac{b}{2a}\right)^2$ is nonnegative, the minimum will occur when this quantity equals 0.
In a similar way, if $a < 0$, the function attains its maximum when $x = -\dfrac{b}{2a}$.
For $f(x) = x^2 - 2mx + 8m + 4$, the minimum occurs when $x = -\dfrac{-2m}{2} = m$. The minimum value of $f(x)$ is therefore

$$f(m) = m^2 - 2m(m) + 8m + 4 = -m^2 + 8m + 4.$$

This is a quadratic function in the variable m with a negative leading coefficient.
Therefore, the maximum occurs when $m = -\dfrac{8}{2(-1)} = 4$.
The maximum of these minimum values is $-4^2 + 8(4) + 4 = 20$.

ANSWER: 20

18. We will require the following fact: If a and r are real numbers with $-1 < r < 1$, then the sum of the *geometric series*

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

is $\dfrac{a}{1-r}$.
We will compute this sum by finding the sums $t_1 + t_3 + t_5 + \cdots$ and $t_2 + t_4 + t_6 + \cdots$ and adding the results. For the first sum, we have

$$\frac{1}{7} + \frac{1}{7^3} + \frac{1}{7^5} + \cdots$$

which can be rewritten as

$$\frac{1}{7} + \frac{1}{7}\left(\frac{1}{49}\right) + \frac{1}{7}\left(\frac{1}{49}\right)^2 + \frac{1}{7}\left(\frac{1}{49}\right)^3 + \cdots$$

since $7^2 = 49$.
This is a geometric series with $a = \dfrac{1}{7}$ and $r = \dfrac{1}{49}$. We also have that $-1 < \dfrac{1}{49} < 1$, so the sum of the series above is

$$\frac{\frac{1}{7}}{1 - \frac{1}{49}} = \frac{\frac{49}{7}}{49 - 1} = \frac{7}{48}.$$

We now compute $t_2 + t_4 + t_6 + \cdots$. Similar to before, we can write the sum as

$$\frac{2}{49} + \frac{2}{49}\left(\frac{1}{49}\right) + \frac{2}{49}\left(\frac{1}{49}\right)^2 + \frac{2}{49}\left(\frac{1}{49}\right)^3 + \cdots$$

which is a geometric series with $a = \frac{2}{49}$ and $r = \frac{1}{49}$. Thus, its sum is

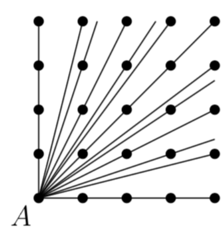
$$\frac{\frac{2}{49}}{1 - \frac{1}{49}} = \frac{2}{49 - 1} = \frac{2}{48}.$$

The sum $t_1 + t_2 + t_3 + \cdots$ is therefore $\frac{7}{48} + \frac{2}{48} = \frac{9}{48} = \frac{3}{16}$.

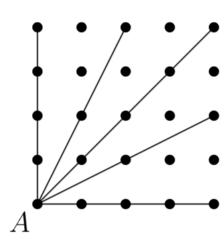
ANSWER: $\frac{3}{16}$

19. *In this solution, we will use the notation* $\binom{n}{2}$ *for "n choose 2", which denotes the number of ways of choosing* 2 *distinct objects from a set of* n *distinct objects. The quantity can be computed as* $\binom{n}{2} = \frac{n(n-1)}{2}$. *The reason for this is that we can choose the "first" object in* n *ways, and then there are* $n - 1$ *ways to choose the "second" object from those remaining. This gives* $n(n - 1)$ *ways of choosing the two objects. However, we do not care about order, so we divide by* 2 *since each pair of objects has been counted twice.*

Out of the 24 points different from A, we need to choose 2. There are $\binom{24}{2}$ ways to do this. Using the formula before the solution, there are $\frac{24(23)}{2} = 12 \times 23 = 276$ choose two points different from A.
Two points will form a triangle together with point A unless all three points lie on a common line.
Therefore, we can get the answer by counting the number of ways to choose two points so that they and A all lie on the same line, and then subtract this number from 276.
We now draw every line through A and some other point in the diagram. Note that many of these lines will be the same. For example, drawing a line through A and any of the other four points in the bottom row gives the same line regardless of which of the four points is chosen.
There are 13 lines in total:



Eight of these lines pass through only one of the points other than A, so there cannot be a pair of points different from A on such a line.
Eliminating these lines from the diagram, we have



There are five lines remaining. Two of them contain exactly two points other than A, so each of these identifies one pair of points that, together with A, do not form a triangle.
Each of the other three remaining lines contains four points other than A, so each of these three lines identifies $\binom{4}{2} = 6$ pairs of points that, together with A, do not form a triangle.
Therefore, there are $276 - 2(1) - 3(6) = 256$ ways to choose two points from the array that, together with A, form a triangle.

ANSWER: 256

20. We will imagine that the track has been broken into 12 equal units.
With this convention, Isabelle will begin to run once Lyla has ran 4 units, and Isabelle will always run 5 units in the amount of time it takes Lyla to run 4 units.
When Lyla has ran a total of 4 units, Isabelle has ran 0 units. At this time, Isabelle starts to run, so when Lyla has ran $4 + 4 = 8$ units, Isabelle will have ran 5 units.
When Lyla has ran $8 + 4 = 12$ units, Isabelle will have ran $5 + 5 = 10$ units.
When Lyla has ran $12 + 4 = 16$ units, Isabelle will have ran $10 + 5 = 15$ units.
When Lyla has ran $16 + 4 = 20$ units, Isabelle will have ran $15 + 5 = 20$ units.
At this point, Isabelle passes Lyla for the first time.
Since Isabelle runs faster than Lyla, Isabelle will always have ran farther than Lyla from this point on.
In the time it takes Lyla to run 12 units, Isabelle will run 15 units.
Continuing the calculation above, when Lyla has ran $20 + 12 = 32$ units, Isabelle will have ran $20 + 15 = 35$ units.
When Lyla has ran $32 + 12 = 44$ units, Isabelle will have ran $35 + 15 = 50$ units.
When Lyla has ran $44 + 12 = 56$ units, Isabelle will have ran $50 + 15 = 65$ units.
When Lyla has ran $56 + 12 = 68$ units, Isabelle will have ran $65 + 15 = 80$ units.

At this point, Isabelle is exactly 12 units, or one lap, ahead of Lyla.
Similarly, at any time when Isabelle passes Lyla, the difference between the number of units they have ran will be a multiple of 12.
This means the 5th time Isabelle passes Lyla will occur when Isabelle has gained an additional $3 \times 12 = 36$ units on Lyla.
Isabelle gains one unit on Lyla for every 4 units Lyla runs, which means that Isabelle will gain 36 units on Lyla when Lyla runs $4 \times 36 = 144$ units.
Therefore, when Isabelle passes Lyla for the 5th time, Lyla has ran a total of $68 + 144 = 212$ units.
There are 12 units in a lap, so this means Lyla will have ran $\frac{212}{12} = 17\frac{8}{12}$ laps.
When Isabelle passes Lyla for the 5th time, Lyla will have completed 17 laps.

ANSWER: 17

21. Using a logarithm rule, we have that $\log x^2 = 2\log x$, so

$$f(x) = 2\sin^2\big(\log x\big) + \cos\big(2\log x\big) - 5.$$

Using the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$, we get

$$f(x) = 2\sin^2(\log x) + \cos^2(\log x) - \sin^2(\log x) - 5$$

which simplifies to

$$f(x) = \sin^2(\log x) + \cos^2(\log x) - 5.$$

Finally, using the fact that $\sin^2 u + \cos^2 u = 1$ for any real number u, we have

$$f(x) = 1 - 5 = -4$$

and this holds for all $x > 0$. Therefore, $f(\pi) = -4$.

ANSWER: -4

22. An important observation for this problem is that every integer has a remainder of either 0, 1, or 2 when it is divided by 3. That is, every integer is either a multiple of 3, one more than a multiple of 3, or two more than a multiple of 3. Suppose $x + y + z$ is a multiple of 3.
If x is a multiple of 3, then $y + z$ is also a multiple of 3. There are three ways this can happen: both y and z are multiples of 3, y is one more than a multiple of 3 and z is two more than a multiple of 3, or y is two more than a multiple of 3 and z is one more than a multiple of 3.
If x is one more than a multiple of 3, then $y + z$ must be two more than a multiple of 3. Again, there are three ways that this can happen.
We summarize the possible remainders after x, y, and z are divided by 3 (either 0, 1, or 2) that will lead to $x + y + z$ being a multiple of 3:

x	y	z
0	0	0
0	1	2
0	2	1
1	0	2
1	1	1
1	2	0
2	0	1
2	1	0
2	2	2

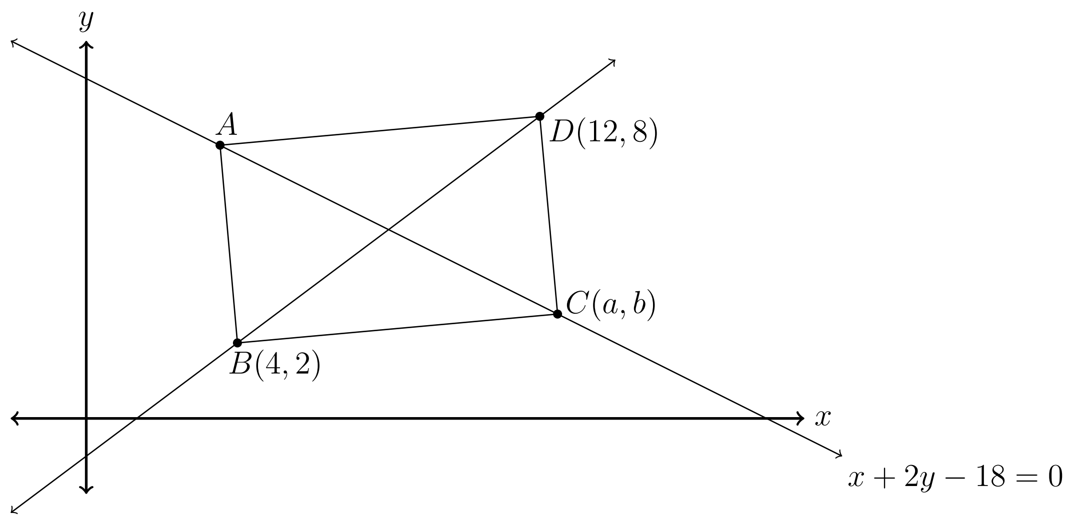
Of the integers from 1 to 10 inclusive, 3, 6, and 9 are multiples of 3 (a total of three), 1, 4, 7, and 10 are one more than a multiple of 3 (a total of 4), and 2, 5, and 8 are two more than a multiple of 3 (a total of 3).
Thus, for example, there are $3 \times 3 \times 3 = 27$ triples (x, y, z) so that $x + y + z$ is a multiple of 3 and each of x, y, and z is a multiple of 3.
Computing the total for each row in the table above, we have

x	y	z	Number of Triples
0	0	0	$3 \times 3 \times 3 = 27$
0	1	2	$3 \times 4 \times 3 = 36$
0	2	1	$3 \times 3 \times 4 = 36$
1	0	2	$4 \times 3 \times 3 = 36$
1	1	1	$4 \times 4 \times 4 = 64$
1	2	0	$4 \times 3 \times 3 = 36$
2	0	1	$3 \times 3 \times 4 = 36$
2	1	0	$3 \times 4 \times 3 = 36$
2	2	2	$3 \times 3 \times 3 = 27$

which means the total is $2(27) + 64 + 6(36) = 54 + 64 + 216 = 334$.

ANSWER: 334

23. Suppose C has coordinates (a, b) and is the vertex of the rectangle to the right of B and on the line $x + 2y - 18 = 0$ as pictured below.



By finding the coordinates of C, we will be able to compute the lengths of BC and CD and use them to find the area of the rectangle.
First, we find the equation of the line through B and D.
Since this line passes through the points $(4, 2)$ and $(12, 8)$, it has slope $\frac{8-2}{12-4} = \frac{6}{8} = \frac{3}{4}$.
Thus, its equation is of the form $y = \frac{3}{4}x + k$ for some k.
Substituting $(x, y) = (4, 2)$, we have $2 = \frac{3}{4}(4) + k$ so $k = -1$, which means the equation of the line is $y = \frac{3}{4}x - 1$.
To find the coordinates of C, we will use that the point of intersection of the diagonals of a rectangle is equidistant from all four of its vertices.
The intersection of the diagonals is the intersection of the lines with equations $y = \frac{3}{4}x - 1$ and $x + 2y - 18 = 0$.
Substituting, we have $x + 2\left(\frac{3}{4}x - 1\right) = 18$ or $x + \frac{3}{2}x - 2 = 18$. Multiplying through by 2 and rearranging, we have $2x + 3x = 40$, so $x = 8$ from which we get $y = \frac{3}{4}(8) - 1 = 5$.
The coordinates of the centre of the rectangle are $\left(\frac{4+12}{2}, \frac{2+8}{2}\right) = (8, 5)$.
The distance from this point to point B is

$$\sqrt{(8-4)^2 + (5-2)^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

Therefore, C is the point on the line $x + 2y = 18$ that is 5 units away from $(8, 5)$.
The condition on the distance gives us $\sqrt{(a-8)^2 + (b-5)^2} = 5$ or $(a-8)^2 + (b-5)^2 = 25$.
Since (a, b) is on $x + 2y = 18$, we also have that $a + 2b = 18$ which can be rearranged to get $a - 8 = 10 - 2b = 2(5 - b)$.
Substituting $a - 8 = 2(5 - b)$ into $(a-8)^2 + (b-5)^2 = 5$ gives

$$[2(5-b)]^2 + (b-5)^2 = 25$$

which simplifies to $5(b-5)^2 = 25$ or $(b-5)^2 = 5$.
Thus, $(b-5) = \pm 5$, so $b = 5 \pm \sqrt{5}$. The point C is to the right of the centre of the rectangle, and since the line connecting A and C has negative slope, this means b is less than 5.
Therefore, $b = 5-\sqrt{5}$. Substituting this into $a+2b = 18$, we have $a = 18-2(5-\sqrt{5}) = 8+2\sqrt{5}$.

Using the formula for the distance between two points in the plane, we have

$$\begin{aligned} BC &= \sqrt{(4-a)^2 + (2-b)^2} \\ &= \sqrt{\left(4-(8+2\sqrt{5})\right)^2 + \left(2-(5-\sqrt{5})\right)} \\ &= \sqrt{(-4-2\sqrt{5})^2 + (-3+\sqrt{5})^2} \\ &= \sqrt{16+16\sqrt{5}+20+9-6\sqrt{5}+5} \\ &= \sqrt{50+10\sqrt{5}}. \end{aligned}$$

Similarly, we compute the length of CD

$$\begin{aligned} CD &= \sqrt{(a-12)^2 + (b-8)^2} \\ &= \sqrt{\left(8+2\sqrt{5}-12\right)^2 + \left(5-\sqrt{5}-8\right)^2} \\ &= \sqrt{(-4+2\sqrt{5})^2 + (-3-\sqrt{5})^2} \\ &= \sqrt{16-16\sqrt{5}+20+9+6\sqrt{5}+5} \\ &= \sqrt{50-10\sqrt{5}}. \end{aligned}$$

Finally, we compute the area of the rectangle

$$\begin{aligned} BC \times CD &= \sqrt{50+10\sqrt{5}} \times \sqrt{50-10\sqrt{5}} \\ &= \sqrt{\left(50+10\sqrt{5}\right)\left(50-10\sqrt{5}\right)} \\ &= \sqrt{50^2 - \left(10\sqrt{5}\right)^2} \\ &= \sqrt{2500-500} \\ &= \sqrt{2000}. \end{aligned}$$

So the area of the rectangle is $\sqrt{2000}$, and since $2000 = 20^2 \times 5$, this simplifies to $20\sqrt{5}$.

ANSWER: $20\sqrt{5}$

24. By clearing denominators, the equation becomes

$$2x(x^2+x+3) + 3x(x^2+5x+3) = (x^2+5x+3)(x^2+x+3)$$

and after expanding the left side, we get

$$2x^3 + 2x^2 + 6x + 3x^3 + 15x^2 + 9x = (x^2+5x+3)(x^2+x+3).$$

There are many ways to systematically expand the right side. One way is to think of it as a binomial

$$\begin{aligned}(x^2+5x+3)(x^2+x+3) &= \big((x^2+(5x+3)\big)\big(x^2+(x+3)\big)\\ &= x^4+x^2(x+3)+(5x+3)x^2+(5x+3)(x+3)\\ &= x^4+x^3+3x^2+5x^3+3x^2+5x^2+15x+3x+9\\ &= x^4+6x^3+11x^2+18x+9.\end{aligned}$$

The original equation is equivalent to

$$2x^3+2x^2+6x+3x^3+15x^2+9x = x^4+6x^3+11x^2+18x+9$$

which can be rearranged to

$$x^4+x^3-6x^2+3x+9=0.$$

Generally, finding the real roots of a quartic polynomial is quite difficult, but the *rational roots theorem* tells us that any rational roots of the quartic above must be divisors of 9. *In general, there could be real roots that are not rational, but that will not be the case here.*
The possible rational roots of the quartic are ±1, ±3, and ±9.
Checking $x=1$, we find that $(1)^4+(1)^3-6(1)^2+3(1)+9=8$, so 1 is not a root.
Checking $x=-1$, we find that $(-1)^4+(-1)^3-6(-1)^2+3(-1)+9=1-1-6-3+9=0$ so -1 is a root.
This means $x+1$ is a factor of the quartic. After factoring, we get that

$$x^4+x^3-6x^2+3x+9=(x+1)(x^3-6x+9).$$

We now try to find a rational root of x^3-6x+9. Again, the possibilities are ±1, ±3, and ±9. Of these six possibilities, only -3 is a root:

$$(-3)^3-6(-3)+9=-27+18+9=0.$$

This means $x+3$ s a factor of x^3-6x+9, and after factoring, we get that

$$x^4+x^3-6x^2+3x+9=(x+1)(x+3)(x^2-3x+3).$$

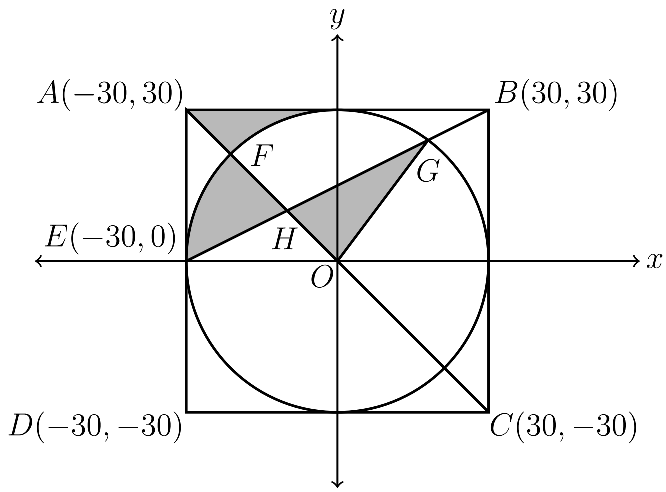
Thus, the equation we wish to solve is equivalent to

$$(x+3)(x+1)(x^2-3x+3)=0.$$

The roots of the quartic are therefore $x=-1$ and $x=-3$ and the roots of x^2-3x+3.
Using the quadratic formula, the two roots of x^2-3x+3 are $\dfrac{3+\sqrt{-3}}{2}$ and $\dfrac{3-\sqrt{-3}}{2}$, neither of which are real.
Therefore, the only real values of x that can possibly satisfy the original equation are -3 and -1.
It can be checked that both $x=-1$ and $x=-3$ satisfy the original equation, so the answer is $(-1)+(-3)=-4$.

Answer: -4

25. We will suppose O is at the origin, which gives rise to coordinates $A(-30, 30)$, $B(30, 30)$, $C(30, -30)$, and $D(-30, -30)$. Additionally, we will have $E(-30, 0)$.



The line passing through A and C also passes through the origin and has slope $m = \dfrac{-30-30}{30-(-30)} = \dfrac{-60}{60} = -1$, and hence has equation $y = -x$.

The line through B and E has slope $m = \dfrac{30-0}{30-(-30)} = \dfrac{30}{60} = \dfrac{1}{2}$ and passes through $(30, 30)$, so its equation is $y = \dfrac{1}{2}x + b$ for some real number b.

By substituting $(x, y) = (30, 30)$, we have $30 = \dfrac{1}{2}(30) + b$ or $b = 15$. Thus, the line passing through both B and E has equation $y = \dfrac{1}{2}x + 15$.

We can now compute the coordinates of points G and H.

Since H the intersection of line segments AC and BE, the x-coordinate of H must satisfy $-x = \dfrac{1}{2}x + 15$ or $-15 = \dfrac{3}{2}x$, which can be solved to get $x = -10$.

Since H lies on the line $y = -x$, this means H has coordinates $(-10, 10)$.

The circle has diameter equal to the side length of the square, which is given to be 60. Therefore, its radius is $\dfrac{60}{2} = 30$, which means the equation of the circle is $x^2 + y^2 = 30^2$.

Points G and E are the points of intersection with the circle and the line $y = \dfrac{1}{2}x + 15$, which means their x-coordinates can be found by substituting $\dfrac{1}{2}x + 15$ in the equation of the circle for y and solving for x.

That is, the x-coordinates of E and G are the solutions to the equation

$$x^2 + \left(\frac{1}{2}x + 15\right)^2 = 900$$

which can be expanded to get

$$x^2 + \frac{1}{4}x^2 + 15x + 225 = 900.$$

Multiplying through by 4 and rearranging, we get

$$5x^2 + 60x - 2700 = 0$$

and after dividing through by 5 we have

$$x^2 + 12x - 540 = 0.$$

We know that the x-coordinate of E is a root, which means $x + 30$ must be a factor of the polynomial on the right, which leads to

$$(x + 30)(x - 18) = 0.$$

Thus, the other solution is $x = 18$, so the x-coordinate of G must be 18. Substituting into $y = \frac{1}{2}x + 15$, we get that $y = \frac{18}{2} + 15 = 24$, so G has coordinates $(18, 24)$.

We next observe that the shaded region bound by the circle and the line segments AF and AB is congruent to the region bound by the circle, AF, and AE.

Thus, the shaded region is equal to the sum of the areas of $\triangle HGO$ and $\triangle AHE$.

If we take AE as the base of $\triangle AHE$, then its height will be the horizontal distance from H to line segment AE. This is equal to the difference between the x coordinates of A and H, or $-10 - (-30) = 20$.

Thus, the area of $\triangle AHE$ is

$$\frac{1}{2} \times AE \times 20 = 10AE = 10(30) = 300.$$

To compute the area of $\triangle HGO$, we will subtract the area of $\triangle EHO$ from the area of $\triangle EGO$. Taking EO as a common base, the heights of $\triangle EHO$ and $\triangle EGO$ are the y-coordinates of H and G, respectively. These values are 10 and 24.

The length of EO is $0 - (-30) = 30$, so the area of $\triangle HGO$ is

$$\begin{aligned}\frac{1}{2} \times EO \times 24 - \frac{1}{2} \times EO \times 10 &= \frac{1}{2} \times EO \times (24 - 10) \\ &= \frac{1}{2}(30)(14) \\ &= 210.\end{aligned}$$

Therefore, the total area of the shaded regions is $300 + 210 = 510$.

Answer: 510

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{2+5\times 5}{3} = \dfrac{2+25}{3} = \dfrac{27}{3} = 9$.

   (b) The area of a triangle with base $2t$ and height $2t-6$ is $\frac{1}{2}(2t)(2t-6)$ or $t(2t-6)$.
   The answer to (a) is 9, so $t=9$ which means $t(2t-6) = 9(12) = 108$.

   (c) Since $\triangle ABC$ is isosceles with $AB = BC$, it is also true that $\angle BCA = \angle BAC$.
   The angles in a triangle add to 180°, so

$$\begin{aligned}180^\circ &= \angle ABC + \angle BAC + \angle BCA \\ &= \angle ABC + 2\angle BAC \\ &= t^\circ + 2\angle BAC\end{aligned}$$

   The answer to (b) is 108, so $t = 108$. Therefore,

$$\angle BAC = \frac{1}{2}(180^\circ - t^\circ) = \frac{1}{2}(180^\circ - 108^\circ) = \frac{1}{2}(72^\circ) = 36^\circ.$$

ANSWER: $9, 108, 36^\circ$

1. (a) In an equilateral triangle, all sides are of equal length. This means $x+5 = 14$ and $y+11 = 14$.
   Solving these equations for x and y, respectively, we get $x = 14-5 = 9$ and $y = 14-11 = 3$.
   Therefore, $x + y = 9 + 3 = 12$.

   (b) Let k be the number of \$1 coins that Gray has. It is given that k is also the number of \$2 coins, which means the total amount of money in dollars that Gray has is

$$k(1) + k(2) = k(1+2) = k(3).$$

   The answer to (a) is 12. Therefore, $\$3k = \$t$ so $k = \dfrac{t}{3} = \dfrac{12}{3} = 4$.

   (c) *Solution 1*
   In the beginning, Elise has tx apples.
   After giving away 10% of her apples, she still had 90% of the original amount, or $\dfrac{9}{10}tx$ apples.
   She gave 6 of these apples to her sister and was left with 48 apples. This gives the equation $0.9tx - 6 = 48$ or $0.9tx = 54$.
   Multiplying both sides by $\dfrac{10}{9}$ gives

$$\frac{10}{9}\left(\frac{9}{10}tx\right) = \frac{10}{9}(54) = 60$$

   so $tx = 60$ which means $x = \dfrac{60}{t}$.
   The answer to part (b) is 4, so $t = 4$ and we have $x = \dfrac{60}{4} = 15$.

Solution 2

Elise had 48 apples after giving 6 to her sister, so she had $48 + 6 = 54$ apples just before she gave apples to her sister.

This means 54 is 90% of the original number of apples.

Multiplying by $\frac{10}{9}$, this means the original number of apples was $\frac{10}{9}(54) = 60$.

The answer to part (b) is 4, so $t = 4$ which means there were four boxes.

There were 60 apples in total and four boxes of apples, which means there were $\frac{60}{4} = 15$ apples in each box.

ANSWER: $12, 4, 15$

2. (a) There are four numbers in total, so their average is

$$\frac{(x+5)+14+x+5}{4} = \frac{2x+24}{4} = \frac{x+12}{2}.$$

The average is 9, so $\frac{x+12}{2} = 9$ or $x + 12 = 18$ which means $x = 6$.

(b) Adding the equations $x+ty+8 = 0$ and $5x-ty+4 = 0$, we get $x+ty+8+5x-ty+4 = 0+0$ or $6x + 12 = 0$.

Solving for x gives $x = -2$. Note that we have found the value of x without knowing the value of t.

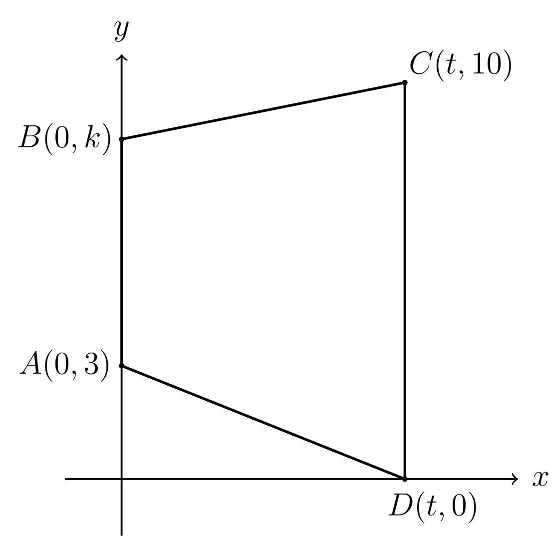
Substituting into the first equation gives $-2+ty+8 = 0$ or $ty = -6$, which means $y = -\frac{6}{t}$.

Substituting the values of x and y into the third equation and rearranging, we have

$$\begin{aligned} 3(-2) - k\left(-\frac{6}{t}\right) + 1 &= 0 \\ -6 + \frac{6k}{t} &= -1 \\ \frac{6k}{t} &= 5 \\ k &= \frac{5t}{6}. \end{aligned}$$

The answer in part (a) is 6, so we have that $t = 6$. Therefore, $k = \frac{5 \times 6}{6} = 5$.

(c) Using the information given, the quadrilateral looks like



We note that the diagram has been drawn with $k < 10$, but this is not necessarily the case.
The sides AB and CD are both vertical because the x-coordinates of the points on each line are constant.
Therefore, AB and CD are parallel, so the quadrilateral is a trapezoid with parallel sides $CD = 10$, $AB = k - 3$, and height t. We are given that $k > 3$, so $k - 3$ is positive and makes sense as a length.
The area of the trapezoid is

$$\frac{t}{2}(AB + CD) = \frac{t}{2}(k - 3 + 10) = \frac{t}{2}(k + 7).$$

Since the area is 50, this means $50 = \frac{t}{2}(k + 7)$ or $k + 7 = \frac{100}{t}$ so $k = \frac{100}{t} - 7$.
The answer to part (b) is 5, so $t = 5$. This means $k = \frac{100}{5} - 7 = 20 - 7 = 13$.

ANSWER: $6, 5, 13$

3. (a) Since $2020 = 5 \times 404$, the positive multiples of 5 less than or equal to 2020 are

$$5 \times 1,\ 5 \times 2,\ 5 \times 3,\ \ldots,\ 5 \times 403,\ 5 \times 404$$

so there are 404 positive multiples of 5 less than or equal to 2020. This means $M = 404$.
Similarly, the number of multiples of 2020 between 1 and 2020 is $N = \frac{2020}{20} = 101$.
Thus, $10M \div N = 4040 \div 101 = 40$.

(b) By opposite angles, $\angle BAC = t^\circ$. This means $\angle BCA = 180^\circ - 2t^\circ - t^\circ = 180^\circ - 3t^\circ$.
Again by opposite angles, $\angle DCE = \angle BCA$, so

$$\begin{aligned} x^\circ &= 180^\circ - \angle CDE - \angle DCE \\ &= 180^\circ - 90^\circ - (180^\circ - 3t^\circ) \\ &= 3t^\circ - 90^\circ. \end{aligned}$$

The answer to part (b) is 40, so $t = 40$ and $x = 3t - 90 = 3 \times 40 - 90 = 30$.

(c) Let A be the number of adults and C be the number of children.
The information given translates into the two equations: $A + C = t$ and $9A + 5C = 190$.
Rearranging the first equation, we have $A = t - C$.
Substituting this into the second equation gives $9(t - C) + 5C = 190$ or $4C = 9t - 190$.
Solving for C, we have $C = \frac{9t - 190}{4}$.
The answer to part (b) is 30, so $t = 30$ and $C = \frac{9(30) - 190}{4} = \frac{80}{4} = 20$.

ANSWER: 40,30,20