The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 18, 2020
(in North America and South America)

Thursday, November 19, 2020
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

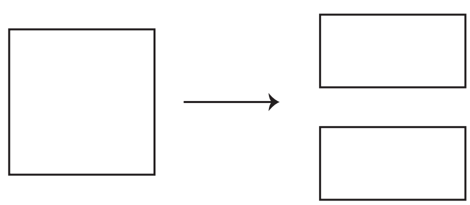
PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

Useful Fact for Part A:
$\sin(x + y) = \sin x \cos y + \cos x \sin y$ for all angles x and y.

1. Markus has 9 candies and Katharina has 5 candies. Sanjiv gives away a total of 10 candies to Markus and Katharina so that Markus and Katharina each end up with the same total number of candies. How many candies does Markus have now?

2. A square is cut into two identical rectangles, as shown.



   Each of these two rectangles has perimeter 24 cm. What is the area of the original square?

3. Suppose that a, b, c, d and e are consecutive positive integers with $a < b < c < d < e$. If $a^2 + b^2 + c^2 = d^2 + e^2$, what is the value of a?

4. Let $\lfloor x \rfloor$ denote the greatest integer which is less than or equal to x. For example, $\lfloor \pi \rfloor = 3$. S is the integer equal to the sum of the 100 terms shown:

$$S = \lfloor \pi \rfloor + \lfloor \pi + \tfrac{1}{100} \rfloor + \lfloor \pi + \tfrac{2}{100} \rfloor + \lfloor \pi + \tfrac{3}{100} \rfloor + \cdots + \lfloor \pi + \tfrac{99}{100} \rfloor$$

   What is the value of S?

5. Suppose that x and y satisfy the equations

$$3\sin x + 4\cos y = 5$$
$$4\sin y + 3\cos x = 2$$

What is the value of $\sin(x+y)$?

6. Suppose that $f(x)$ is a function defined for every real number x with $0 \le x \le 1$ with the properties that

- $f(1-x) = 1 - f(x)$ for all real numbers x with $0 \le x \le 1$,
- $f(\frac{1}{3}x) = \frac{1}{2}f(x)$ for all real numbers x with $0 \le x \le 1$, and
- $f(a) \le f(b)$ for all real numbers $0 \le a \le b \le 1$.

What is the value of $f\left(\frac{6}{7}\right)$?

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

Useful Facts for Part B:
$\gcd(a, bc) = \gcd(a, b)$ for all integers a, b and c with $\gcd(a, c) = 1$.
$\gcd(a, b) = \gcd(a, b - qa)$ for all integers a, b and q.

1. (a) Determine the point of intersection of the lines with equations $y = 4x - 32$ and $y = -6x + 8$.

   (b) Suppose that a is an integer. Determine the point of intersection of the lines with equations $y = -x + 3$ and $y = 2x - 3a^2$. (The coordinates of this point will be in terms of a.)

   (c) Suppose that c is an integer. Show that the lines with equations $y = -c^2x + 3$ and $y = x - 3c^2$ intersect at a point with integer coordinates.

   (d) Determine the four integers d for which the lines with equations $y = dx + 4$ and $y = 2dx + 2$ intersect at a point with integer coordinates.

2. Suppose that $ABCDEF$ is a regular hexagon with sides of length 6. Each interior angle of $ABCDEF$ is equal to 120°.

   (a) A circular arc with centre D and radius 6 is drawn from C to E, as shown. Determine the area of the shaded sector.

   (b) A circular arc with centre D and radius 6 is drawn from C to E, as shown. A second arc with centre A and radius 6 is drawn from B to F, as shown. These arcs are tangent (that is, touch) at the centre of the hexagon. Line segments BF and CE are also drawn. Determine the total area of the shaded regions.

   (c) Along each edge of the hexagon, a semi-circle with diameter 6 is drawn. Determine the total area of the shaded regions; that is, determine the total area of the regions that lie inside exactly two of the semi-circles.

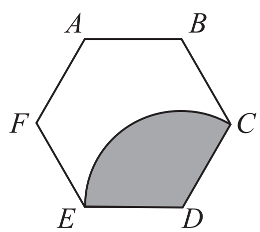


Diagram for (a)

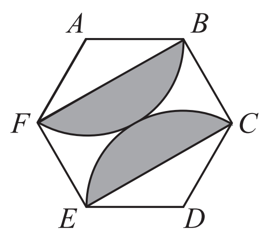


Diagram for (b)

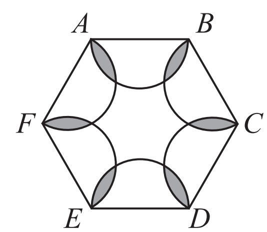


Diagram for (c)

3. Suppose that $f(x) = x^3 - px^2 + qx$ and $g(x) = 3x^2 - 2px + q$ for some positive integers p and q.

   (a) If $p = 33$ and $q = 216$, show that the equation $f(x) = 0$ has three distinct integer solutions and the equation $g(x) = 0$ has two distinct integer solutions.

   (b) Suppose that the equation $f(x) = 0$ has three distinct integer solutions and the equation $g(x) = 0$ has two distinct integer solutions. Prove that

      (i) p must be a multiple of 3,

      (ii) q must be a multiple of 9,

      (iii) $p^2 - 3q$ must be a positive perfect square, and

      (iv) $p^2 - 4q$ must be a positive perfect square.

   (c) Prove that there are infinitely many pairs of positive integers (p, q) for which the following three statements are all true:

      - The equation $f(x) = 0$ has three distinct integer solutions.
      - The equation $g(x) = 0$ has two distinct integer solutions.
      - The greatest common divisor of p and q is 3 (that is, $\gcd(p, q) = 3$).

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(English)