The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

Canadian Intermediate Mathematics Contest

Wednesday, November 18, 2020
(in North America and South America)

Thursday, November 19, 2020
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Intermediate Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

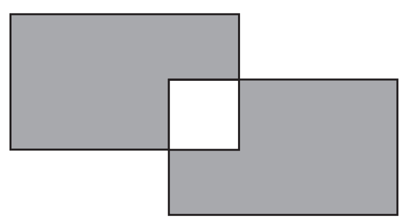
For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

Useful Fact for Part A:
The sum of the measures of the interior angles of a regular polygon with n sides is $180^\circ(n-2)$.

1. The five numbers $\frac{1}{4}, \frac{4}{10}, \frac{41}{100}$, 0.04, 0.404 are to be listed from smallest to largest.

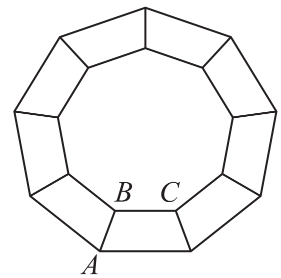
   Which number will be in the middle of the list?

2. In the diagram, two 8 by 10 rectangles overlap to form a 4 by 4 square. What is the total area of the shaded region?



3. A dish contains 100 candies. Juan removes candies from the dish each day and no candies are added to the dish. On day 1, Juan removes 1 candy. On day 2, Juan removes 2 candies. On each day that follows, Juan removes 1 more candy than he removed on the previous day. After day n, Juan has removed a total of at least 64 candies. What is the smallest possible value of n?

4. In the diagram, nine identical isosceles trapezoids are connected as shown to form a closed ring. (An *isosceles trapezoid* is a trapezoid with the properties that one pair of sides is parallel and the other two sides are equal in length.) What is the measure of $\angle ABC$?



5. Determine all pairs (x, y) of integers with $x \leq y$ for which $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{4}$.

6. There are 90 players in a league. Each of the 90 players plays exactly one match against each of the other 89 players. Each match ends with a win for one player and a loss for the other player, or with a tie for both players. Each player earns 1 point for a win, 0 points for a loss, and 0.5 points for a tie. After all matches have been played, the points earned by each player are added up. What is the greatest possible number of players whose total score can be at least 54 points?

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

> *Useful Fact for Part B*:
> The volume of a square-based pyramid whose base has side length a and whose height is h is equal to $\frac{1}{3}a^2h$.

1. An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence with five terms.

   (a) The first four terms of an arithmetic sequence are 20, 13, 6, -1. What are the next two terms in the sequence?

   (b) The numbers 2, a, b, c, 14 in that order form an arithmetic sequence with five terms. What are the values of a, b, and c?

   (c) The numbers 7, 15 and t, arranged in some unknown order, form an arithmetic sequence with three terms. Determine all possible values of t.

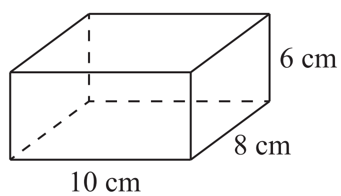
   (d) The numbers r, s, w, x, y, z in that order form an arithmetic sequence with six terms. The sequence includes the numbers 4 and 20, with the number 4 appearing before the number 20 in the sequence. Determine the largest possible value of $z - r$ and the smallest possible value of $z - r$.

2. (a) Tank A and Tank B are rectangular prisms and are sitting on a flat table.
Tank A is 10 cm $\times$ 8 cm $\times$ 6 cm and is sitting on one of its 10 cm $\times$ 8 cm faces.
Tank B is 5 cm $\times$ 9 cm $\times$ 8 cm and is sitting on one of its 5 cm $\times$ 9 cm faces.
Initially, Tank A is full of water and Tank B is empty.
The water in Tank A drains out at a constant rate of 4 cm^3/s.
Tank B fills with water at a constant rate of 4 cm^3/s.
Tank A begins to drain at the same time that Tank B begins to fill.

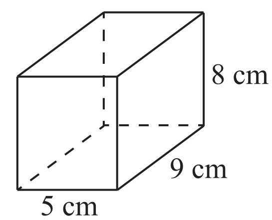
(i) Determine after how many seconds Tank B will be exactly $\frac{1}{3}$ full.

(ii) Determine the depth of the water left in Tank A at the instant when Tank B is full.

(iii) At one instant, the depth of the water in Tank A is equal to the depth of the water in Tank B. Determine this depth.

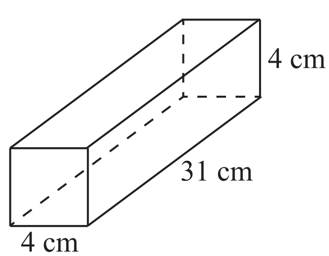


Tank A

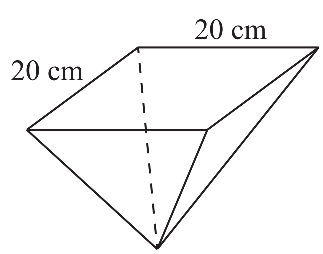


Tank B

(b) Tank C is a rectangular prism that is 31 cm $\times$ 4 cm $\times$ 4 cm.
Tank C sits on the flat table on one of its 31 cm $\times$ 4 cm faces.
Tank D is in the shape of an inverted square-based pyramid, as shown. It is supported so that its square base is parallel to the flat table and its fifth vertex touches the flat table.
The height of Tank D is 10 cm and the side length of its square base is 20 cm.
Initially, Tank C is full of water and Tank D is empty.
Tank D begins filling with water at a rate of 1 cm^3/s.
Two seconds after Tank D begins to fill, Tank C begins to drain at a rate of 2 cm^3/s.
At one instant, the volume of water in Tank C is equal to the volume of water in Tank D.
Determine the depth of the water in Tank D at that instant.



Tank C



Tank D

3. The integers a, b, c, and d are each equal to one of 1, 2, 3, 4, 5, 6, 7, 8, 9. (It is possible that two or more of the integers a, b, c, and d have the same value.) The integer P equals the product of a, b, c, and d; that is, $P = abcd$.

   (a) Determine whether or not P can be a multiple of 216.

   (b) Determine whether or not P can be a multiple of 2000.

   (c) Determine the number of possible different values of P that are divisible by 128 but are not divisible by 1024.

   (d) Determine the number of ordered quadruples (a, b, c, d) for which P is 98 less than a multiple of 100.