The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2019 Pascal Contest

(Grade 9)

Tuesday, February 26, 2019
(in North America and South America)

Wednesday, February 27, 2019
(outside of North America and South America)

Solutions

1. Evaluating, $2 \times 3 + 2 \times 3 = 6 + 6 = 12$.

Answer: (D)

2. Since a square has four equal sides, the side length of a square equals one-quarter of the perimeter of the square.
Thus, the side length of a square with perimeter 28 is $28 \div 4 = 7$.

Answer: (E)

3. In the diagram, there are 9 hexagons of which 5 are shaded.
Therefore, the fraction of all of the hexagons that are shaded is $\frac{5}{9}$.

Answer: (B)

4. Since 38% of students received a muffin, then $100\% - 38\% = 62\%$ of students did not receive a muffin.
Alternatively, using the percentages of students who received yogurt, fruit or a granola bar, we see that $10\% + 27\% + 25\% = 62\%$ did not receive a muffin.

Answer: (D)

5. We know that $\frac{1}{2} = 0.5$. Since $\frac{4}{9} \approx 0.44$ is less than $\frac{1}{2} = 0.5$, then 4 cannot be placed in the box. (No integer smaller than 4 can be placed in the box either.) Since $\frac{5}{9} \approx 0.56$ is greater than $\frac{1}{2} = 0.5$, then the smallest integer that can be placed in the box is 5.

Answer: (D)

6. Since $4x + 14 = 8x - 48$, then $14 + 48 = 8x - 4x$ or $62 = 4x$.
Dividing both sides of this equation by 2, we obtain $\frac{4x}{2} = \frac{62}{2}$ which gives $2x = 31$.

Answer: (B)

7. The segment of the number line between 3 and 33 has length $33 - 3 = 30$.
Since this segment is divided into six equal parts, then each part has length $30 \div 6 = 5$.
The segment PS is made up of 3 of these equal parts, and so has length $3 \times 5 = 15$.
The segment TV is made up of 2 of these equal parts, and so has length $2 \times 5 = 10$.
Thus, the sum of the lengths of PS and TV is $15 + 10$ or 25.

Answer: (A)

8. Since $\frac{20}{19}$ is larger than 1 and smaller than 2, and $20 \times 19 = 380$, then $\frac{20}{19} < 20 \times 19 < 2019$.
We note that $19^{20} > 10^{20} > 10\,000$ and $20^{19} > 10^{19} > 10\,000$.
This means that both 19^{20} and 20^{19} are greater than 2019.
In other words, of the five numbers $19^{20}, \frac{20}{19}, 20^{19}, 2019, 20 \times 19$, the third largest is 2019.
Since the list contains 5 numbers, then its median is the third largest number, which is 2019.
(Note that it does not matter whether 19^{20} is greater than or less than 20^{19}.)

Answer: (D)

9. Since the complete angle at the centre of each circle is 360° and the unshaded sector of each circle has central angle 90°, then the unshaded sector of each circle represents $\dfrac{90^\circ}{360^\circ} = \dfrac{1}{4}$ of its area.
In other words, each of the circles is $\dfrac{3}{4}$ shaded.
There are 12 circles in the diagram.
Since the radius of each circle is 1, then the area of each circle is $\pi \times 1^2$ or π.
Therefore, the total shaded area is $\dfrac{3}{4} \times 12 \times \pi$ or 9π.
Answer: (D)

10. Suppose that sixty $1 \times 1 \times 1$ cubes are joined face to face in a single row on a table.
Each of the 60 cubes has its front, top and back faces exposed.
The leftmost and rightmost cubes also have their left and right faces, respectively, exposed.
No other faces are exposed.
Therefore, the number of 1×1 faces that are exposed is $60 \times 3 + 2$ which equals 182.
Answer: (C)

11. Using the second row, we see that the sum of the numbers in each row, column and diagonal must be $3.6 + 3 + 2.4 = 9$.
Since the sum of the numbers in the first column must be 9, then the bottom left number must be $9 - 2.3 - 3.6 = 9 - 5.9 = 3.1$
Since the sum of the numbers in the top left to bottom right diagonal must be 9, then the bottom right number must be $9 - 2.3 - 3 = 9 - 5.3 = 3.7$
Since the sum of the numbers in the bottom row must be 9, then $3.1 + x + 3.7 = 9$ and so $x + 6.8 = 9$ or $x = 9 - 6.8 = 2.2$.
We can complete the magic square as shown:

2.3	3.8	2.9
3.6	3	2.4
3.1	2.2	3.7

Answer: (E)

12. Since $\triangle PQX$ is right-angled at Q, then

$$\angle PXQ = 90^\circ - \angle QPX = 90^\circ - 62^\circ = 28^\circ$$

Since $\angle PXQ$ and $\angle SXR$ are opposite, then $\angle SXR = \angle PXQ = 28^\circ$.
Since $\triangle RXS$ is isosceles with $RX = SX$, then $\angle XRS = \angle XSR = y^\circ$.
Since the angles in any triangle have a sum of 180°,

$$\begin{aligned} \angle XRS + \angle XSR + \angle SXR &= 180^\circ \\ y^\circ + y^\circ + 28^\circ &= 180^\circ \\ 2y + 28 &= 180 \\ 2y &= 152 \end{aligned}$$

and so $y = 76$.
Answer: (C)

13. *Solution 1*
Since p, q, r, s is a list of consecutive integers in increasing order, then q is 1 more than p and r is 1 less than s.
This means that $q + r = (p + 1) + (s - 1) = p + s = 109$.

Solution 2
Since p, q, r, s is a list of consecutive integers in increasing order, then $q = p+1$, $r = p+2$, and $s = p + 3$.
Since $p + s = 109$, then $p + p + 3 = 109$ and so $2p = 106$ or $p = 53$.
This means that $q = 54$ and $r = 55$. Thus, $q + r = 109$.

Answer: (B)

14. Since the ratio of the number of skateboards to the number of bicycles was $7 : 4$, then the numbers of skateboards and bicycles can be written in the form $7k$ and $4k$ for some positive integer k.
Since the difference between the numbers of skateboards and bicycles is 12, then $7k - 4k = 12$ and so $3k = 12$ or $k = 4$.
Therefore, the total number of skateboards and bicycles is $7k + 4k = 11k = 11 \times 4 = 44$.

Answer: (A)

15. For Sophie's average over 5 tests to be 80%, the sum of her marks on the 5 tests must be $5 \times 80\% = 400\%$.
After the first 3 tests, the sum of her marks is $73\% + 82\% + 85\% = 240\%$.
Therefore, she will reach her goal as long as the sum of her marks on the two remaining tests is *at least* $400\% - 240\% = 160\%$.
The sums of the pairs of marks given are (A) 161%, (B) 161%, (C) 162%, (D) 156%, (E) 160%.
Thus, the pair with which Sophie would not meet her goal is (D).

Answer: (D)

16. *Solution 1*
Since the result must be the same for any real number x less than -2, we substitute $x = -4$ into each of the five expressions:

(A) $x = -4$ (B) $x + 2 = -2$ (C) $\frac{1}{2}x = -2$ (D) $x - 2 = -6$ (E) $2x = -8$

Therefore, $2x$ is the expression with the least value when $x = -4$ and thus must always be the expression with the least value.

Solution 2
For any real number x, we know that $x - 2$ is less than x which is less than $x + 2$.
Therefore, neither x nor $x + 2$ can be the least of the five values.
For any negative real number x, the value of $2x$ will be less than the value of $\frac{1}{2}x$.
Therefore, $\frac{1}{2}x$ cannot be the least of the five values.
Thus, the least of the five values is either $x - 2$ or $2x$.
When $x < -2$, we know that $2x - (x - 2) = x + 2 < 0$.
Since the difference between $2x$ and $x - 2$ is negative, then $2x$ has the smaller value and so is the least of all five values.

Answer: (E)

17. Each of the animals is either striped or spotted, but not both.
Since there are 100 animals and 62 are spotted, then there are $100 - 62 = 38$ striped animals.
Each striped animal must have wings or a horn, but not both.
Since there are 28 striped animals with wings, then there are $38 - 28 = 10$ striped animals with horns.
Each animal with a horn must be either striped or spotted.
Since there are 36 animals with horns, then there are $36 - 10 = 26$ spotted animals with horns.

ANSWER: (E)

18. By the Pythagorean Theorem,

$$QT^2 = QP^2 + PT^2 = k^2 + k^2 = 2k^2$$

Since $QT > 0$, then $QT = \sqrt{2}k$.
Since $\triangle QTS$ is isosceles, then $TS = QT = \sqrt{2}k$.
By the Pythagorean Theorem,

$$QS^2 = QT^2 + TS^2 = (\sqrt{2}k)^2 + (\sqrt{2}k)^2 = 2k^2 + 2k^2 = 4k^2$$

Since $QS > 0$, then $QS = 2k$.
Since $\triangle QSR$ is isosceles, then $SR = QS = 2k$.
Since $\triangle QPT$ is right-angled at P, its area is $\frac{1}{2}(QP)(PT) = \frac{1}{2}k^2$.
Since $\triangle QTS$ is right-angled at T, its area is $\frac{1}{2}(QT)(TS) = \frac{1}{2}(\sqrt{2}k)(\sqrt{2}k) = \frac{1}{2}(2k^2) = k^2$.
Since $\triangle QSR$ is right-angled at S, its area is $\frac{1}{2}(QS)(SR) = \frac{1}{2}(2k)(2k) = 2k^2$.
Since the sum of the three areas is 56, then $\frac{1}{2}k^2 + k^2 + 2k^2 = 56$ or $\frac{7}{2}k^2 = 56$ which gives $k^2 = 16$.
Since $k > 0$, then $k = 4$.

ANSWER: (C)

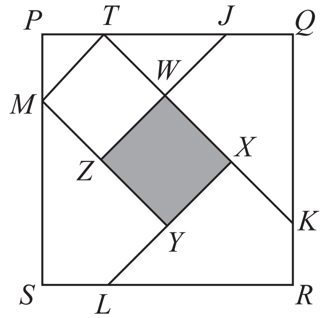
19. Since 4 balls are chosen from 6 red balls and 3 green balls, then the 4 balls could include:

- 4 red balls, or
- 3 red balls and 1 green ball, or
- 2 red balls and 2 green balls, or
- 1 red ball and 3 green balls.

There is only 1 different-looking way to arrange 4 red balls.
There are 4 different-looking ways to arrange 3 red balls and 1 green ball: the green ball can be in the 1st, 2nd, 3rd, or 4th position.
There are 6 different-looking ways to arrange 2 red balls and 2 green balls: the red balls can be in the 1st/2nd, 1st/3rd, 1st/4th, 2nd/3rd, 2nd/4th, or 3rd/4th positions.
There are 4 different-looking ways to arrange 1 red ball and 3 green balls: the red ball can be in the 1st, 2nd, 3rd, or 4th position.
In total, there are $1 + 4 + 6 + 4 = 15$ different-looking arrangements.

ANSWER: (A)

20. Since the sides of quadrilateral $WXYZ$ are parallel to the diagonals of square $PQRS$ and the diagonals of a square are perpendicular, then the sides of $WXYZ$ are themselves perpendicular. This means that quadrilateral $WXYZ$ has four right angles and is thus a rectangle.
Since the diagram does not change when rotated by $90°$, $180°$ or $270°$, then it must be the case that $WX = XY = YZ = YW$ which means that $WXYZ$ is a square.
We calculate the area of $WXYZ$ by first calculating the length of WZ.
Extend KW to meet PQ at T. Join M to T.



Since TK is parallel to diagonal PR, then $\angle QTK = \angle QKT = 45°$, which means that $\triangle TQK$ is isosceles with $QT = QK$.
Since $QR = 40$ and $KR = 10$, then $QK = QR - KR = 30$ and so $QT = 30$.
Since $PQ = 40$, then $PT = PQ - QT = 10$.
Since $PM = PT = 10$, then $\triangle MPT$ is right-angled and isosceles as well, which means that MT is actually parallel to diagonal SQ. (We did not construct MT with this property, but it turns out to be true.)
Since the sides of $MTWZ$ are parallel to the diagonals of the square, then $MTWZ$ is also a rectangle, which means that $WZ = MT$.
Since $PM = PT$, then

$$MT = \sqrt{PM^2 + PT^2} = \sqrt{10^2 + 10^2} = \sqrt{200}$$

by the Pythagorean Theorem.
Thus, $WZ = MT = \sqrt{200}$ and so the area of square $WXYZ$ equals WZ^2 or 200.

Answer: (B)

21. The units digit of 5^{2019} is 5.
This is because the units digit of any power of 5 is 5.
To see this, we note that the first few powers of 5 are

$$5^1 = 5 \quad 5^2 = 25 \quad 5^3 = 125 \quad 5^4 = 625 \quad 5^5 = 3125 \quad 5^6 = 15\,625$$

The units digit of a product of integers depends only on the units digits of the integers being multiplied. Since $5 \times 5 = 25$ which has a units digit of 5, the units digits remains 5 for every power of 5.

The units digit of 3^{2019} is 7.
This is because the units digits of powers of 3 cycle $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
To see this, we note that the first few powers of 3 are

$$3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243 \quad 3^6 = 729$$

Since the units digit of a product of integers depends only on the units digits of the integers being multiplied and we multiply by 3 to get from one power to the next, then once a units digit recurs in the sequence of units digits, the following units digits will follow the same pattern.
Since the units digits of powers of 3 cycle in groups of 4 and 2016 is a multiple of 4, then 3^{2016} has a units digit of 1.
Moving three additional positions along the sequence, the units digit of 3^{2019} will be 7.

Since the units digit of 5^{2019} is 5 and the units digit of 3^{2019} is 7, then the units digit of the difference will be 8. (This is the units digit of the difference whenever a smaller integer with units digit 7 is substracted from a larger integer with units digit 5.)

Answer: (E)

22. *Solution 1*
The smallest integer greater than 2019 that can be formed in this way is formed using the next two largest consecutive integers 20 and 21, giving the four-digit integer 2120.
The largest such integer is 9998.
The list of such integers is

$$2120, 2221, 2322, \ldots, 9796, 9897, 9998$$

Each pair of consecutive numbers in this list differs by 101 since the number of hundreds increases by 1 and the number of ones increases by 1 between each pair.
Since the numbers in the list are equally spaced, then their sum will equal the number of numbers in the list times the average number in the list.
The average number in the list is $\dfrac{2120 + 9998}{2} = \dfrac{12118}{2} = 6059$.
Since each number in the list is 101 greater than the previous number, then the number of increments of 101 from the first number to the last is $\dfrac{9998 - 2120}{101} = \dfrac{7878}{101} = 78$.
Since the number of increments is 78, then the number of numbers is 79.
This means that the sum of the numbers in the list is $79 \times 6059 = 478\,661$.

Solution 2
As in Solution 1, the list of such integers is $2120, 2221, 2322, \ldots, 9796, 9897, 9998$.
If the sum of these integers is S, then

$$\begin{aligned}
S &= 2120 + 2221 + 2322 + \ldots + 9796 + 9897 + 9998 \\
&= (2100 + 2200 + 2300 + \cdots + 9700 + 9800 + 9900) \\
&\qquad + (20 + 21 + 22 + \cdots + 96 + 97 + 98) \\
&= 100(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) \\
&\qquad + (20 + 21 + 22 + \cdots + 96 + 97 + 98) \\
&= 100(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) \\
&\qquad + (21 + 22 + \cdots + 96 + 97 + 98 + 99) + 20 - 99 \\
&= 101(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) - 79
\end{aligned}$$

There are 79 numbers in the list of consecutive integers from 21 to 99, inclusive, and the middle number in this list is 60.
Therefore, $S = 101 \times 79 \times 60 - 79 = 478\,661$.

Answer: (C)

23. Since the wheel turns at a constant speed, then the percentage of time when a shaded part of the wheel touches a shaded part of the path will equal the percentange of the total length of the path where there is "shaded on shaded" contact.
Since the wheel has radius 2 m, then its circumference is $2\pi \times 2$ m which equals 4π m.
Since the wheel is divided into four quarters, then the portion of the circumference taken by each quarter is π m.
We label the left-hand end of the path 0 m.
As the wheel rotates once, the first shaded section of the wheel touches the path between 0 m and π m ≈ 3.14 m.
As the wheel continues to rotate, the second shaded section of the wheel touches the path between 2π m ≈ 6.28 m and 3π m ≈ 9.42 m.
The path is shaded for 1 m starting at each odd multiple of 1 m, and unshaded for 1 m starting at each even multiple of 1 m.
Therefore, the first shaded section touches shaded stripes between 1 m and 2 m, and between 3 m and π m.
The second shaded section touches shaded stripes between 7 m and 8 m, and between 9 m and 3π m.
Therefore, the total length of "shaded on shaded" is 1 m + $(\pi - 3)$ m + 1 m + $(3\pi - 9)$ m or $(4\pi - 10)$ m.
The total length of the path along which the wheel rolls is 4π m.
This means that the required percentage of time equals $\dfrac{(4\pi - 10)\text{ m}}{4\pi\text{ m}} \times 100\% \approx 20.4\%$.
Of the given choices, this is closest to 20%, or choice (A).

Answer: (A)

24. First, we note that 88 663 311 000 is divisible by 792. (We can check this by division.)
Therefore, 88 663 311 000 000 is also divisible by 792.
Since 88 663 311 000 000 is divisible by 792, then 88 663 311 $pqr\,s$48 is divisible by 792 exactly when $pqr\,s$48 is divisible by 792. (This comes from the fact that if the difference between two integers is divisible by d, then either both are divisible by d or neither is divisible by d.)
The smallest integer of the form $pqr\,s$48 is 48 (which is "000 048") and the largest integer of the form $pqr\,s$48 is 999 948.
Since $999\,948 \div 792 \approx 1262.6$, then the multiples of 792 in between 48 and 999 948 are the integers of the form $792 \times n$ where $1 \leq n \leq 1262$.
Suppose that $792 \times n = pqr\,s48$ for some integer n.
Comparing units digits, we see that the units digit of n must be 4 or 9.
This means that $n = 10c + 4$ or $n = 10c + 9$ for some integer $c \geq 0$.

In the first case, $792(10c + 4) = 7920c + 3168$.
This integer has a units digit of 8.
For this integer to have a tens digit of 4, we need $2c+6$ to have a units digit of 4, which happens exactly when c has units digit 4 or 9.
This means that c can be $4, 9, 14, 19, 24, \ldots$.
This means that n can be $44, 94, 144, 194, 244, \ldots$.
Since $1 \leq n \leq 1262$, then there are 25 possible values of n with units digit 4, because there are 2 values of n between 0 and 100, 2 between 100 and 200, and so on up to 1200, with an additional 1 (namely, 1244) between 1200 and 1262.

In the second case, $792(10c + 9) = 7920c + 7128$.
This integer has a units digit of 8.

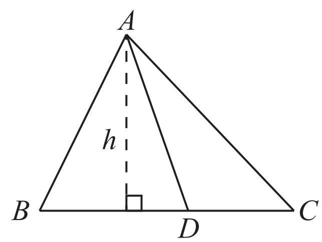
For this integer to have a tens digit of 4, we need $2c+2$ to have units digit 4, which happens exactly when c has units digit 1 or 6.
This means that c can be $1, 6, 11, 16, 21, \ldots$.
This means that n can be $19, 69, 119, 169, 219, \ldots$.
Since $1 \leq n \leq 1262$, then there are again 25 possible values of n with units digit 9.

In total, there are $25+25=50$ 14-digit positive integers of the desired form that are divisible by 792.

ANSWER: (E)

25. In $\triangle ABC$, if D is on BC, then

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{BD}{CD}$$



This is because $\triangle ABD$ and $\triangle ACD$ have a common height of h and so

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} == \frac{\frac{1}{2} \times BD \times h}{\frac{1}{2} \times CD \times h} = \frac{BD}{CD}$$

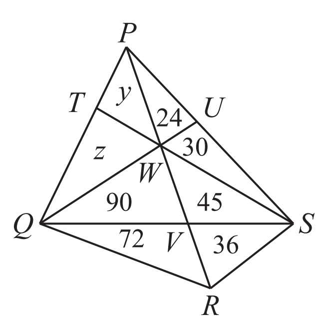
Using this in the given diagram with point V lying on side QS of $\triangle WQS$ and on side QS of $\triangle RQS$, we see that

$$\frac{\text{Area of } \triangle QVW}{\text{Area of } \triangle SVW} = \frac{QV}{SV} = \frac{\text{Area of } \triangle QVR}{\text{Area of } \triangle SVR}$$

Combining the first and third parts of this equality, we obtain the equivalent equations

$$\begin{aligned}
\frac{8x+50}{5x+20} &= \frac{8x+32}{5x+11} \\
(8x+50)(5x+11) &= (8x+32)(5x+20) \\
40x^2+88x+250x+550 &= 40x^2+160x+160x+640 \\
338x+550 &= 320x+640 \\
18x &= 90 \\
x &= 5
\end{aligned}$$

Since $x=5$, then we can calculate and fill in the area of most of the pieces of the diagram:



Let the area of $\triangle PTW$ equal y and the area of $\triangle QTW$ equal z.
We know that
$$\frac{\text{Area of } \triangle QVW}{\text{Area of } \triangle SVW} = \frac{QV}{SV}$$
and so $\dfrac{QV}{SV} = \dfrac{90}{45} = 2$.
Since V lies on side QS of $\triangle PQS$, then
$$\frac{\text{Area of } \triangle PQV}{\text{Area of } \triangle PSV} = \frac{QV}{SV} = 2$$
and so $\dfrac{y+z+90}{99} = 2$ which gives $y+z+90 = 198$ and so $y+z = 108$.
Finally, looking at points W on side TS of $\triangle PTS$ and so side TS of $\triangle QTS$, we get
$$\frac{y}{54} = \frac{TW}{WS} = \frac{z}{135}$$
and so $135y = 54z$ or $5y = 2z$ or $z = \frac{5}{2}y$.
Therefore, $y + \frac{5}{2}y = 108$ or $\frac{7}{2}y = 108$ and so $y = \frac{216}{7} = 30\frac{6}{7}$.
Of the given choices, the area of $\triangle PTW$ is closest to 31.

Answer: (E)