The CENTRE for EDUCATION
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2019 Hypatia Contest

Wednesday, April 10, 2019
(in North America and South America)

Thursday, April 11, 2019
(outside of North America and South America)

Solutions

1. (a) The radius of each hole is 2 cm, and so the diameter of each hole is 4 cm.
Since there are 4 holes, then their diameters combine for a total distance of $4 \times 4 = 16$ cm along the 91 cm midline.
Thus, the 5 equal spaces along the midline combine for a total distance of $91 - 16 = 75$ cm, and so the distance along the midline between adjacent holes is $\frac{75}{5} = 15$ cm.

(b) Let the radius of each hole be r cm, and so the diameter of each hole is $2r$ cm.
Since there are 4 holes, then their diameters combine for a total distance of $4 \times 2r = 8r$ cm along the midline.
The distance along the midline between adjacent holes is equal to the radius, r cm, and so these 5 equal spaces along the midline combine for a total distance of $5r$ cm.
The total length of the midline is 91 cm, and so $8r + 5r = 91$ or $13r = 91$, and so the radius of each hole is $\frac{91}{13} = 7$ cm.

(c) *Solution 1*
As in part (b), if the diameter of each hole is $2r$ cm, then the 4 diameters combine for a total distance of $4 \times 2r = 8r$ cm along the midline.
If the distance along the midline between adjacent holes is 5 cm, then the 5 equal spaces along the midline combine for a total distance of 25 cm.
The total length of the midline is 91 cm, and so we get $8r + 25 = 91$ or $8r = 66$, and so the radius of each hole is $\frac{66}{8} = 8.25$ cm.
However, the vertical distance from the midline to each edge of the metal is 8 cm, and since the holes must be circles, the radius of each hole cannot be 8.25 cm, and so the distance between adjacent holes cannot be 5 cm.

Solution 2
The minimum possible distance between adjacent holes is determined by maximizing the radius of each of the circles.
Since the holes must be circles, and the vertical distance from the midline to each edge of the metal is 8 cm, the maximum radius is 8 cm.
Since there are 4 holes, then their diameters combine for a maximum total distance of $4 \times 16 = 64$ cm along the midline.
The total length of the midline is 91 cm, and so the 5 equal spaces combine for a minimum total length of $91 - 64 = 27$ cm.
Thus, the minimum distance between adjacent holes is $\frac{27}{5} = 5.4$ cm, which is greater than the required 5 cm.

2. (a) To add a bump, the line segment of length 21 is first broken into three segments, each having length $\frac{21}{3} = 7$.
The middle segment of these three segments is removed, and two new segments each having length 7 are added.
Thus, after a bump is added to a segment of length 21, the new path will have length $4 \times 7 = 28$.

(b) A path with exactly one bump has four line segments of equal length.
If such a path has length 240, then each of the four line segments has length $\frac{240}{4} = 60$.
Thus, the original line segment had three line segments each of length 60, and so the length of the original line segment was $3 \times 60 = 180$.

(c) To add the first bump, the line segment of length 36 is broken into three segments, each having length $\frac{36}{3} = 12$.
The middle segment of these three segments is removed, and two new segments each having length 12 are added.
Thus, after the first bump is added to a segment of length 36, the new path will have length $4 \times 12 = 48$.

Next, a bump is added to each of the four segments having length 12.
Consider adding a bump to one of these four segments.
The segment of length 12 is broken into three segments, each having length $\frac{12}{3} = 4$.
The middle segment is removed, two new segments each having length 4 are added, and so the new length is $4 \times 4 = 16$.
There are four such segments to which this process happens, and so the total path length of the resulting figure is $4 \times 16 = 64$.

(d) To add a bump, the line segment of length n is first broken into three segments, each having length $\frac{n}{3}$.
The middle segment of these three segments is removed, and two new segments each having length $\frac{n}{3}$ are added.
Thus, after a bump is added to a segment of length n, Path 1 will have length $4 \times \frac{n}{3} = \frac{4}{3}n$.

To create Path 2, a bump is added to each of the four segments having length $\frac{n}{3}$.
Consider adding a bump to one of these four segments.
The segment of length $\frac{n}{3}$ is broken into three segments, each having length $\frac{1}{3} \times \frac{n}{3}$.
The middle segment is removed, two new segments each having length $\frac{1}{3} \times \frac{n}{3}$ are added, and so the new length is $4 \times \frac{1}{3} \times \frac{n}{3} = \frac{4}{3^2}n$.
There are four such segments to which this process happens, and so the total length of Path 2 is $4 \times \frac{4}{3^2}n = \frac{4^2}{3^2}n$ or $\left(\frac{4}{3}\right)^2 n$.

To summarize, when a bump is added to a line segment of length n, the length of the resulting path, Path 1, is $\frac{4}{3}n$.

When bumps are then added to Path 1, the length of the resulting Path 2 is $\left(\frac{4}{3}\right)^2 n$.

This process will continue with each new path having a total length that is $\frac{4}{3}$ of the previous path length.
That is, Path 3 will have length $\left(\frac{4}{3}\right)^3 n$, Path 4 will have length $\left(\frac{4}{3}\right)^4 n$, and Path 5 will have length $\left(\frac{4}{3}\right)^5 n$.
If the length of Path 5 is an integer, then n must be divisible by 3^5 (since there are no factors of 3 in 4^5).
The smallest integer n which is divisible by 3^5 is $3^5 = 243$.
The smallest possible integer n for which the length of Path 5 is an integer is 243.

3. (a) The arithmetic mean of 36 and 64 is $\dfrac{36+64}{2} = \dfrac{100}{2} = 50$.

The geometric mean of 36 and 64 is $\sqrt{36 \cdot 64} = \sqrt{6^2 \cdot 8^2} = 6 \cdot 8 = 48$.

(b) If the arithmetic mean of two positive real numbers x and y is 13, then $\dfrac{x+y}{2} = 13$.
If the geometric mean of two positive real numbers x and y is 12, then $\sqrt{xy} = 12$.
Multiplying the first equation by 2 gives $x + y = 26$ and so $x = 26 - y$.
Substituting $x = 26 - y$ into the second equation and squaring both sides gives $(26 - y)y = 12^2$ or $y^2 - 26y + 144 = 0$.
Factoring the left side of this equation, we get $(y - 8)(y - 18) = 0$ and so $y = 8$ or $y = 18$.
When $y = 8$, $x = 26 - 8 = 18$, and when $y = 18$, $x = 8$.
That is, the numbers 8 and 18 have an arithmetic mean of 13 and a geometric mean of 12.

(c) We are required to solve the equation $\dfrac{x+y}{2} - \sqrt{xy} = 1$, for positive integers x and y with $x < y \le 50$.
Simplifying first, we get

$$\begin{aligned}
\frac{x+y}{2} - \sqrt{xy} &= 1 \\
x - 2\sqrt{xy} + y &= 2 \\
(\sqrt{x})^2 - 2\sqrt{xy} + (\sqrt{y})^2 &= 2 \quad \text{(since } x, y > 0\text{)} \\
(\sqrt{x} - \sqrt{y})^2 &= 2 \\
\sqrt{x} - \sqrt{y} &= \pm\sqrt{2} \\
\sqrt{y} - \sqrt{x} &= \sqrt{2} \quad \text{(since } y > x\text{)} \\
\sqrt{y} &= \sqrt{x} + \sqrt{2} \\
y &= x + 2 + 2\sqrt{2x}
\end{aligned}$$

Since x is a positive integer, then $y = 2 + x + 2\sqrt{2x}$ is a positive integer exactly when $\sqrt{2x}$ is a positive integer.
This occurs exactly when $x = 2m^2$ for integer values of m.
Since $x < y \le 50$, then $2m^2 < 50$ or $m^2 < 25$, and so m is any integer from 1 to 4 inclusive.
We determine the corresponding values of x and y in the table below.

m	$x = 2m^2$	$y = 2 + x + 2\sqrt{2x}$
1	2	8
2	8	18
3	18	32
4	32	50

The pairs (x, y) satisfying the required conditions are $(2, 8), (8, 18), (18, 32)$, and $(32, 50)$.

4. (a) We begin by multiplying the first equation by 5 and the second equation by 3 to get

$$\begin{aligned}
15x + 20y &= 50 \\
15x + 18y &= 3c
\end{aligned}$$

By subtracting the second equation from the first, we get

$$(15x + 20y) - (15x + 18y) = 50 - 3c$$

or $2y = 50 - 3c$. Solving for y, we get

$$y = 25 - \frac{3}{2}c$$

Substituting this into the first of the original two equations, we get

$$3x + 4\left(25 - \frac{3}{2}c\right) = 10$$

Multiplying the 4 through the parentheses gives $3x + 100 - 6c = 10$ which simplifies to $3x = 6c - 90$ or $x = 2c - 30$. Therefore, in terms of c, we have

$$(x, y) = \left(2c - 30, 25 - \frac{3}{2}c\right)$$

(b) Similar to part (a), we will first solve for x and y in terms of d.
This will give us enough information to determine for which d these values of x and y are integers.
Multiplying the first equation by 4 we get $4x + 8y = 12$.
We can subtract $4x + dy = 6$ from $4x + 8y = 12$ to get $8y - dy = (8 - d)y = 6$.
This means $y = \dfrac{6}{8-d}$.
From the first equation, we get $x = 3 - 2y$.
Substituting the expression for y into this equation and simplifying, we get

$$x = 3 - 2\left(\frac{6}{8-d}\right) = 3 - \frac{12}{8-d}$$

We need to find values of d so that $y = \dfrac{6}{8-d}$ and $x = 3 - \dfrac{12}{8-d}$ are both integers.
Since 3 is an integer, x will be an integer exactly when $\dfrac{12}{8-d}$ is an integer.
For this to happen, we need $8 - d$ to be a divisor of 12.
However, for $y = \dfrac{6}{8-d}$ to be an integer, we need $8 - d$ to be a divisor of 6.
Since any divisor of 6 is also a divisor of 12, this means that if y is an integer, then x is an integer.
Therefore, we need only find integers d so that y is an integer.
The divisors of 6, and hence, the possible values of $8 - d$, are $-6, -3, -2, -1, 1, 2, 3$, and 6.
Therefore, the possible values of d are $2, 5, 6, 7, 9, 10, 11$, and 14.
(We may check that each of these values of d give integer values for x and y.)

(c) In order to simplify things, we will first show that, regardless of the values of x, y and k, if n is an integer, then y must be equal to -1.
To see this, we begin by multiplying the first equation by -2 and the second equation by 3 to get

$$\begin{aligned} -(18n + 12)x + (6n + 4)y &= -6n^2 - 12n - (6k + 10) \\ (18n + 12)x + (9n^2 + 6n)y &= -3n^2 + (6k + 6) \end{aligned}$$

We now add the two equations.
When doing this, the $-(18n + 12)$ cancels with the $(18n + 12)$ to give

$$(9n^2 + 12n + 4)y = -9n^2 - 12n - 4$$

Observe that $(3n+2)^2 = 9n^2 + 12n + 4$, so we can rewrite this equation as

$$y(3n+2)^2 = -(3n+2)^2$$

Suppose $(3n+2)^2 = 0$. Then $3n+2 = 0$ so $n = -\frac{2}{3}$, which is not an integer.
Therefore, if we assume n is an integer, we can divide the above equation by $(3n+2)^2$ to get

$$y = \frac{-(3n+2)^2}{(3n+2)^2} = -1$$

We are interested in finding positive integer values of k for which there exist integers n so that the system of equations has a solution (x, y) where x and y are integers.
Thus, going forward, we assume n is an integer which we have shown means $y = -1$.
The first equation simplifies to

$$(9n+6)x - (3n+2)(-1) = 3n^2 + 6n + (3k+5)$$

which can be rearranged to get

$$(9n+6)x = 3n^2 + 3n + 3k + 3 \qquad (1)$$

The second equation becomes

$$(6n+4)x + (3n^2+2n)(-1) = -n^2 + (2k+2)$$

which rearranges to

$$(6n+4)x = 2n^2 + 2n + 2k + 2 \qquad (2)$$

Dividing equation (1) by 3 or equation (2) by 2 gives

$$(3n+2)x = n^2 + n + 1 + k$$

We want n, k, and x to all be integers, which means we need integers n and k so that $n^2 + n + 1 + k$ is a multiple of $3n+2$.
To simplify this, we will show that $n^2 + n + 1 + k$ is a multiple of $3n+2$ if and only if $3(n^2 + n + 1 + k)$ is a multiple of $3n+2$.
Assuming $n^2 + n + 1 + k$ is a multiple of $3n+2$, it is certainly true that $3(n^2 + n + 1 + k)$ is a multiple of $3n+2$.
Notice that $3n+2$ is 2 more than a multiple of 3, so $3n+2$ is not a multiple of 3.
This means $3n+2$ does not have a prime factor of 3, so $3(n^2+n+k+1)$ being a multiple of $3n+2$ means that $n^2 + n + 1 + k$ is a multiple of $3n+2$.
To summarize, we want to understand integer pairs (n, k) for which $n^2 + n + 1 + k$ is a multiple of $3n+2$.
We have shown that finding such integer pairs is the same as finding integer pairs (n, k) for which $3n^2 + 3n + 3 + 3k$ is a multiple of $3n+2$.
By rearranging and factoring, this is the same as finding integer pairs (n, k) so that $n(3n+2) + n + 3 + 3k$ is a multiple of $3n+2$.
Noticing that

$$\frac{n(3n+2) + n + 3 + 3k}{3n+2} = n + \frac{n+3+3k}{3n+2}$$

the expression on the right side is an integer if and only if $n+3+3k$ is a multiple of $3n+2$.
Therefore, we need to find integer pairs (n, k) such that $n+3+3k$ is a multiple of $3n+2$.
Using the same reasoning as before, since $3n+2$ does not have a prime factor of 3, we have that $n+3+3k$ is a multiple of $3n+2$ if and only if $3(n+3+3k) = 3n+9+9k$ is a multiple of $3n+2$.
Notice that

$$\frac{3n+9+9k}{3n+2} = \frac{(3n+2)+(7+9k)}{3n+2} = 1 + \frac{7+9k}{3n+2}$$

so $3n+9+9k$ is a multiple of $3n+2$ if and only if $\dfrac{7+9k}{3n+2}$ is an integer, or $7+9k$ is a multiple of $3n+2$.
Putting all of this together, given that n and k are integers, the system of equations has a solution (x, y) where x and y are both integers precisely when $7+9k$ is a multiple of $3n+2$.
Phrasing the question in these terms, we are looking for a positive integer k for which there are exactly eight integers n so that $3n+2$ is a factor of $9k+7$.
This means we need a positive integer k so that exactly eight of the factors of $9k+7$ are two more than a multiple of 3.
Beginning with $k=1$, we get $9k+7=16$.
The factors of 16 are $-16, -8, -4, -2, -1, 1, 2, 4, 8$, and 16, of which only $-16, -4, -1, 2$, and 8 are 2 more than a multiple of 3.
When $k=2$, $9k+7=25$.
The factors of 25 are $-25, -5, -1, 1, 5$, and 25.
There are fewer than eight factors in total, so $k=2$ does not work.
When $k=3$, $9k+7=34$.
The factors of 34 are $-34, -17, -2, -1, 1, 2, 17, 34$, of which only $-34, -1, 2$, and 17 are 2 more than a multiple of 3.
When $k=4$, $9k+7=43$ which is prime, so it only has four factors in total which means $k=4$ does not work.
When $k=5$, $9k+7=52$.
The factors of 52 are $-52, -26, -13, -4, -2, -1, 1, 2, 4, 13, 26,$ and 52, of which only $-52, -13, -4, -1, 2$, and 26 are 2 more than a multiple of 3.
When $k=6$, $9k+7=61$ which is prime, so it only has four factors in total which means $k=6$ does not work.
When $k=7$, $9k+7=70$.
The factors of 70 are

$$-70, -35, -14, -10, -7, -5, -2, -1, 1, 2, 5, 7, 10, 14, 35, \text{ and } 70$$

Of these, the numbers $-70, -10, -7, -1, 2, 5, 14$ and 35 are two more than a multiple of 3.
There are exactly 8 numbers in this list.
Therefore, if $k=7$, there are exactly eight integers n $(-70, -10, -7, -1, 2, 5, 14$, and $35)$, with the property that the system of equations has a solution (x, y) where x and y are integers.
(It is worth noting that there are other values of k which satisfy the given properties.)