Galois Contest

(Grade 10)

Wednesday, April 10, 2019
(in North America and South America)

Thursday, April 11, 2019
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes　　　　　　　　　　　　　　　　　　©2019 University of Waterloo

Do not open this booklet until instructed to do so.

Number of questions: 4　　　　　　　　**Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 💡

   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

2. **FULL SOLUTION** parts indicated by ✎

   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions must be supplied by your supervising teacher and inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked 💡, place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked ✎, provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

1. The Galois Restaurant is in a region that adds 10% sales tax onto the price of food and drinks purchased at a restaurant. The prices listed on their menu do not include the sales tax.

   💡 (a) From the menu, Becky orders a plate of lasagna listed for $7.50, a side salad listed for $5.00, and a lemonade listed for $3.00. After tax is included, how much is Becky's total bill?

   💡 (b) A burrito is listed on the menu for $6.00. After tax is included, what is the greatest number of burritos that Jackson can buy if he has $50.00?

   ✎ (c) On the Galois Restaurant menu, hotdogs are listed at the regular price of $5.00. The restaurant has the following promotional deals:

   • On Mondays, if you buy a hotdog at the regular menu price of $5.00, then the price for a second hotdog is $4.50.

   • On Tuesdays, you pay half the tax on all hotdogs.

   Chase bought two hotdogs on Monday and then two hotdogs on Tuesday. After tax is included, determine on which day Chase spent less money.

2. The hypotenuse of right-angled $\triangle AOB$ lies on the line with equation $y = -2x + 12$, as shown in Figure 1. The legs of $\triangle AOB$ lie on the axes.

    (a) What is the area of $\triangle AOB$?

    (b) A second line passes through O and is perpendicular to the first line, as shown in Figure 2. The two lines intersect at C. Determine the coordinates of C.

    (c) The second line passes through the point D in the first quadrant, as shown in Figure 3. Points E and F are positioned on the axes so that $DEOF$ is a rectangle. If the area of $DEOF$ is 1352, determine the coordinates of D.
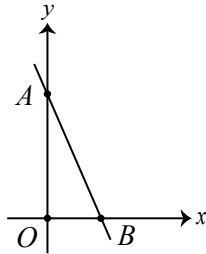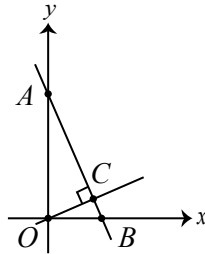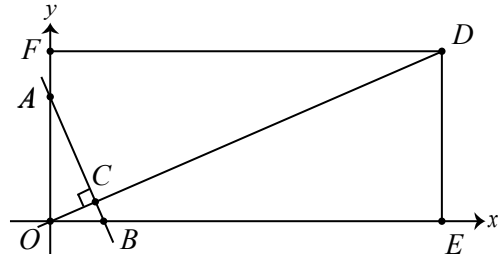

Figure 1   Figure 2   Figure 3

3. If n is a positive integer, the notation $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.

    (a) What is the largest positive integer m for which 2^m is a divisor of $9!$?

    (b) What is the smallest value of n for which $n!$ is divisible by 7^2?

    (c) Explain why there is no positive integer n for which $n!$ is divisible by 7^7 but is not divisible by 7^8.

    (d) Show that there is exactly one positive integer n for which

$$n! = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23, \text{ and}$$

$$a + b + c + d = 45$$

for some positive integers a, b, c, d.

4. A positive integer is *digit-balanced* if each digit d, with $0 \le d \le 9$, appears at most d times in the integer. For example, 13224 is digit-balanced, but 21232 is not.

    (a) Explain why a digit-balanced integer is not divisible by 10.

    (b) How many 4-digit integers have all non-zero digits and are *not* digit-balanced?

    (c) Determine all positive integers k for which there exist digit-balanced positive integers m and n, where $m + n = 10^k$ and m and n each have k digits.

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Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2019.

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For teachers...

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- Register your students for the Canadian Senior and Intermediate Mathematics Contests which will be written in November
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