The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2019 Fermat Contest

(Grade 11)

Tuesday, February 26, 2019
(in North America and South America)

Wednesday, February 27, 2019
(outside of North America and South America)

Solutions

1. Since the largest multiple of 5 less than 14 is 10 and $14 - 10 = 4$, then the remainder when 14 is divided by 5 is 4.

   ANSWER: (E)

2. Simplifying, we see that $20(x + y) - 19(y + x) = 20x + 20y - 19y - 19x = x + y$ for all values of x and y.

   ANSWER: (B)

3. Evaluating, $8 - \dfrac{6}{4 - 2} = 8 - \dfrac{6}{2} = 8 - 3 = 5$.

   ANSWER: (A)

4. The segment of the number line between 3 and 33 has length $33 - 3 = 30$.
   Since this segment is divided into six equal parts, then each part has length $30 \div 6 = 5$.
   The segment PS is made up of 3 of these equal parts, and so has length $3 \times 5 = 15$.
   The segment TV is made up of 2 of these equal parts, and so has length $2 \times 5 = 10$.
   Thus, the sum of the lengths of PS and TV is $15 + 10$ or 25.

   ANSWER: (A)

5. Since 1 hour equals 60 minutes, then 20 minutes equals $\frac{1}{3}$ of an hour.
   Since Mike rides at 30 km/h, then in $\frac{1}{3}$ of an hour, he travels $\frac{1}{3} \times 30$ km $= 10$ km.

   ANSWER: (E)

6. Suppose that $SU = UW = WR = b$ and $PS = h$.
   Since the width of rectangle $PQRS$ is $3b$ and its height is h, then its area is $3bh$.
   Since $SU = b$ and the distance between parallel lines PQ and SR is h, then the area of $\triangle STU$ is $\frac{1}{2}bh$. Similarly, the area of each of $\triangle UVW$ and $\triangle WXR$ is $\frac{1}{2}bh$.
   Therefore, the fraction of the rectangle that is shaded is $\dfrac{3 \times \frac{1}{2}bh}{3bh}$ which equals $\dfrac{1}{2}$.

   ANSWER: (C)

7. Since Cans is north of Ernie, then Ernie cannot be the town that is the most north.
   Since Dundee is south of Cans, then Dundee cannot be the town that is the most north.
   Since Arva is south of Blythe, then Arva cannot be the town that is the most north.
   Since Arva is north of Cans, then Cans cannot be the town that is the most north.
   The only remaining possibility is that Blythe is the town that is the most north.
   The following arrangement is the unique one that satisfies the given conditions:

   Blythe
   Arva
   Cans
   Dundee
   Ernie

   ANSWER: (B)

8. We note that $8 \times 48 \times 81 = 2^3 \times (2^4 \times 3) \times 3^4 = 2^7 \times 3^5 = 2^2 \times 2^5 \times 3^5 = 2^2 \times (2 \times 3)^5 = 2^2 \times 6^5$.
   After 6^5 is divided out from $8 \times 48 \times 81$, the quotient has no factors of 3 and so no further factors of 6 can be divided out.
   Therefore, the largest integer k for which 6^k is a divisor of $8 \times 48 \times 81$ is $k = 5$.

   ANSWER: (C)

9. The average of $\frac{1}{8}$ and $\frac{1}{6}$ is $\dfrac{\frac{1}{8}+\frac{1}{6}}{2} = \dfrac{\frac{3}{24}+\frac{4}{24}}{2} = \frac{1}{2} \times \frac{7}{24} = \frac{7}{48}$.

ANSWER: (E)

10. We find the smallest such integer greater than 30 000 and the largest such integer less than 30 000 and then determine which is closest to 30 000.
Let M be the smallest integer greater than 30 000 that is formed using the digits 2, 3, 5, 7, and 8, each exactly once.
Since M is greater than 30 000, its ten thousands digit is at least 3.
To make M as small as possible (but greater than 30 000), we set its ten thousands digit to 3.
To make M as small as possible, its thousands digit should be as small as possible, and thus equals 2.
Continuing in this way, its hundreds, tens and ones digits are 578. Thus, $M = 32\,578$.
Let m be the largest integer less than 30 000 that is formed using the digits 2, 3, 5, 7, and 8, each exactly once.
Since m is less than 30 000, its ten thousands digit is less than 3 and must thus be 2.
To make m as large as possible (but less than 30 000), its thousands digit should be as large as possible, and thus equals 8.
Continuing in this way, its hundreds, tens and ones digits are 7, 5 and 3, respectively. Thus, $m = 28\,753$.
Since $M - 30\,000 = 2578$ and $30\,000 - m = 1247$, then m is closer to 30 000.
Thus, $N = m = 28\,753$. The tens digit of N is 5.

ANSWER: (B)

11. The line with equation $y = x - 3$ has slope 1.
To find the x-intercept of the line with equation $y = x - 3$, we set $y = 0$ and solve for x to obtain $x - 3 = 0$ or $x = 3$. Thus, line ℓ also has x-intercept 3.
Further, since the two lines are perpendicular, the slopes of the two lines have a product of -1, which means that the slope of ℓ is -1.
Line ℓ has slope -1 and passes through $(3, 0)$.
This means that ℓ has equation $y - 0 = -1(x - 3)$ or $y = -x + 3$.
Therefore, the y-intercept of line ℓ is 3.

ANSWER: (C)

12. Alberto answered 70% of 30 questions correctly in the first part.
Thus, Alberto answered $\frac{70}{100} \times 30 = 21$ questions correctly in the first part.
Alberto answered 40% of 50 questions correctly in the second part.
Thus, Alberto answered $\frac{40}{100} \times 50 = 20$ questions correctly in the second part.
Overall, Alberto answered $21 + 20 = 41$ of $30 + 50 = 80$ questions correctly.
This represents a percentage of $\frac{41}{80} \times 100\% = 51.25\%$.
Of the given choices, this is closest to 51%.

ANSWER: (D)

13. The number of minutes between 7:00 a.m. and the moment when Tanis looked at her watch was $8x$, and the number of minutes between the moment when Tanis looked at her watch and 8:00 a.m. was $7x$.
The total number of minutes between 7:00 a.m. and 8:00 a.m. is 60.
Therefore, $8x + 7x = 60$ and so $15x = 60$ or $x = 4$.
The time at that moment was $8x = 32$ minutes after 7:00 a.m., and so was 7:32 a.m. (We note that 7:32 a.m. is $28 = 7x$ minutes before 8:00 a.m.)

ANSWER: (C)

14. Each letter A, B, C, D, E appears exactly once in each column and each row.
The entry in the first column, second row cannot be A or E or B (the entries already present in that column) and cannot be C or A (the entries already present in that row).
Therefore, the entry in the first column, second row must be D.
This means that the entry in the first column, fourth row must be C.
The entry in the fifth column, second row cannot be D or C or A or E and so must be B.
This means that the entry in the second column, second row must be E.
Using similar arguments, the entries in the first row, third and fourth columns must be D and B, respectively.
This means that the entry in the second column, first row must be C.
Using similar arguments, the entries in the fifth row, second column must be A.
Also, the entry in the third row, second column must be D.
This means that the letter that goes in the square marked with $*$ must be B.
We can complete the grid as follows:

A	C	D	B	E
D	E	C	A	B
E	D	B	C	A
C	B	A	E	D
B	A	E	D	C

Answer: (B)

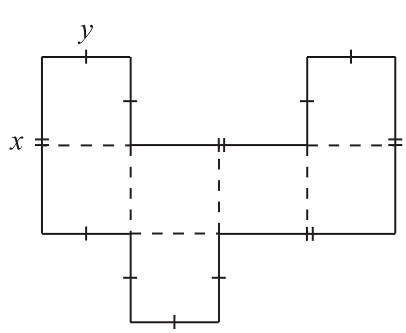
15. Since 4 balls are chosen from 6 red balls and 3 green balls, then the 4 balls could include:

- 4 red balls, or
- 3 red balls and 1 green ball, or
- 2 red balls and 2 green balls, or
- 1 red ball and 3 green balls.

There is only 1 different-looking way to arrange 4 red balls.
There are 4 different-looking ways to arrange 3 red balls and 1 green ball: the green ball can be in the 1st, 2nd, 3rd, or 4th position.
There are 6 different-looking ways to arrange 2 red balls and 2 green balls: the red balls can be in the 1st/2nd, 1st/3rd, 1st/4th, 2nd/3rd, 2nd/4th, or 3rd/4th positions.
There are 4 different-looking ways to arrange 1 red ball and 3 green balls: the red ball can be in the 1st, 2nd, 3rd, or 4th position.
In total, there are $1 + 4 + 6 + 4 = 15$ different-looking arrangements.

Answer: (A)

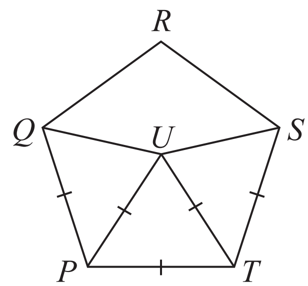
16. Since $x = 2y$, then by drawing dotted lines parallel to the line segments in the given figure, some of which start at midpoints of the current sides, we can divide the figure into 7 squares, each of which is y by y.



Since the area of the given figure is 252, then $7y^2 = 252$ or $y^2 = 36$.
Since $y > 0$, then $y = 6$.
The perimeter of the figure consists of 16 segments of length y.
Therefore, the perimeter is $16 \times 6 = 96$.

Answer: (A)

17. Join QU and SU.
Since $\triangle PUT$ is equilateral, then $PU = UT = TP$.
Since pentagon $PQRST$ is regular, then $QP = PT = TS$.
Thus, $PU = QP$ and $UT = TS$, which means that $\triangle QPU$ and $\triangle STU$ are isosceles.



Each interior angle in a regular pentagon measures 108°.
Since $\angle UPT = 60^\circ$, then $\angle QPU = \angle QPT - \angle UPT = 108^\circ - 60^\circ = 48^\circ$.
Since $\triangle QPU$ is isosceles with $QP = PU$, then $\angle PQU = \angle PUQ$.
Thus, $\angle PUQ = \frac{1}{2}(180^\circ - \angle QPU) = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ$.
By symmetry, $\angle TUS = 66^\circ$.
Finally, $\angle QUS = 360^\circ - \angle PUQ - \angle PUT - \angle TUS = 360^\circ - 66^\circ - 60^\circ - 66^\circ = 168^\circ$.

Answer: (B)

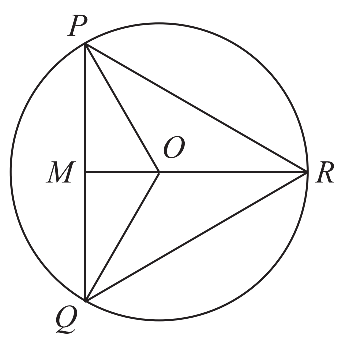
18. Let n be a 7-digit positive integer made up of the digits 0 and 1 only, and that is divisible by 6.
The leftmost digit of n cannot be 0, so must be 1.
Since n is divisible by 6, then n is even, which means that the rightmost digit of n cannot be 1, and so must be 0.
Therefore, n has the form $1\,pqr\,st0$ for some digits p, q, r, s, t each equal to 0 or 1.
n is divisible by 6 exactly when it is divisible by 2 and by 3.
Since the ones digit of n is 0, then it is divisible by 2.
n is divisible by 3 exactly when the sum of its digits is divisible by 3.
The sum of the digits of n is $1 + p + q + r + s + t$.
Since each of p, q, r, s, t is 0 or 1, then $1 \leq 1 + p + q + r + s + t \leq 6$.
Thus, n is divisible by 3 exactly when $1 + p + q + r + s + t$ is equal to 3 or to 6.
That is, n is divisible by 3 exactly when either 2 of p, q, r, s, t are 1s or all 5 of p, q, r, s, t are 1s.
There are 10 ways for 2 of these to be 1s.
These correspond to the pairs $pq, pr, ps, pt, qr, qs, qt, rs, rt, st$.
There is 1 way for all 5 of p, q, r, s, t to be 1s.
Thus, there are $1 + 10 = 11$ such 7-digit integers.

Answer: (B)

19. We use the functional equation $f(2x+1) = 3f(x)$ repeatedly.
Setting $x = 1$, we get $f(3) = 3f(1) = 3 \times 6 = 18$.
Setting $x = 3$, we get $f(7) = 3f(3) = 3 \times 18 = 54$.
Setting $x = 7$, we get $f(15) = 3f(7) = 3 \times 54 = 162$.
Setting $x = 15$, we get $f(31) = 3f(15) = 3 \times 162 = 486$.
Setting $x = 31$, we get $f(63) = 3f(31) = 3 \times 486 = 1458$.

Answer: (D)

20. Suppose that a circle with centre O has radius 2 and that equilateral $\triangle PQR$ has its vertices on the circle.
Join OP, OQ and OR.
Join O to M, the midpoint of PQ.



Since the radius of the circle is 2, then $OP = OQ = OR = 2$.
By symmetry, $\angle POQ = \angle QOR = \angle ROP$.
Since these three angles add to 360°, then $\angle POQ = \angle QOR = \angle ROP = 120^\circ$.
Since $\triangle POQ$ is isosceles with $OP = OQ$ and M is the midpoint of PQ, then OM is an altitude and an angle bisector.
Therefore, $\angle POM = \frac{1}{2}\angle POQ = 60^\circ$ which means that $\triangle POM$ is a 30°-60°-90° triangle.
Since $OP = 2$ and is opposite the 90° angle, then $OM = 1$ and $PM = \sqrt{3}$.
Since $PM = \sqrt{3}$, then $PQ = 2PM = 2\sqrt{3}$.
Therefore, the area of $\triangle POQ$ is $\frac{1}{2} \cdot PQ \cdot OM = \frac{1}{2} \cdot 2\sqrt{3} \cdot 1 = \sqrt{3}$.
Since $\triangle POQ$, $\triangle QOR$ and $\triangle ROP$ are congruent, then they each have the same area.
This means that the area of $\triangle PQR$ is three times the area of $\triangle POQ$, or $3\sqrt{3}$.

Answer: (A)

21. *Solution 1*

We start with the ones digits.
Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.
Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.
Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.
Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.
Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.
Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.
This gives the following completed multiplication:

$$\begin{array}{rr} & 1\,0\,2\,5\,6\,4 \\ \times & 4 \\ \hline & 4\,1\,0\,2\,5\,6 \end{array}$$

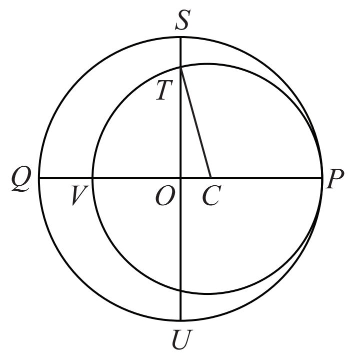
Finally, $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Solution 2

Let x be the five-digit integer with digits $PQRST$.
This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.
Also, $4PQRST = 400\,000 + PQRST = 400\,0000 + x$.
From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives $40x + 16 = 400\,000 + x$ or $39x = 399\,984$.
Thus, $x = \dfrac{399\,984}{39} = 10\,256$.
Since $PQRST = 10\,256$, then $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Answer: (A)

22. Let D be the length of the diameter of the larger circle and let d be the length of the diameter of the smaller circle.
Since QP and VP are diameters of the larger and smaller circles, then $QV = QP - VP = D - d$.
Since $QV = 9$, then $D - d = 9$.
Let C be the centre of the smaller circle and join C to T. Since $D > d$, C is to the right of O along QP.



Since CT is a radius of the smaller circle, then $CT = \frac{1}{2}d$.
Also, $OC = OP - CP$. Since OP and CP are radii of the two circles, then $OC = \frac{1}{2}D - \frac{1}{2}d$.
Since SO is a radius of the larger circle and $ST = 5$, then $TO = SO - ST = \frac{1}{2}D - 5$.
Since QP and SU are perpendicular, then $\triangle TOC$ is right-angled at O.
By the Pythagorean Theorem,

$$\begin{aligned}
TO^2 + OC^2 &= CT^2 \\
\left(\tfrac{1}{2}D - 5\right)^2 + \left(\tfrac{1}{2}D - \tfrac{1}{2}d\right)^2 &= \left(\tfrac{1}{2}d\right)^2 \\
4\left(\tfrac{1}{2}D - 5\right)^2 + 4\left(\tfrac{1}{2}D - \tfrac{1}{2}d\right)^2 &= 4\left(\tfrac{1}{2}d\right)^2 \\
(D - 10)^2 + (D - d)^2 &= d^2 \\
(D - 10)^2 + 9^2 &= d^2 \\
81 &= d^2 - (D - 10)^2 \\
81 &= (d - (D - 10))(d + (D - 10)) \\
81 &= (d - D + 10)(d + (D - 10)) \\
81 &= (10 - (D - d))(d + D - 10) \\
81 &= (10 - 9)(d + D - 10) \\
81 &= d + D - 10 \\
91 &= d + D
\end{aligned}$$

and so the sum of the diameters is 91.

Answer: (B)

23. We consider first the integers that can be expressed as the sum of exactly 4 consecutive positive integers.
The smallest such integer is $1+2+3+4 = 10$. The next smallest such integer is $2+3+4+5 = 14$.
We note that when we move from $k+(k+1)+(k+2)+(k+3)$ to $(k+1)+(k+2)+(k+3)+(k+4)$, we add 4 to the total (this equals the difference between $k+4$ and k since the other three terms do not change).
Therefore, the positive integers that can be expressed as the sum of exactly 4 consecutive positive integers are those integers in the arithemetic sequence with first term 10 and common difference 4.
Since $n \leq 100$, these integers are

$$10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98$$

There are 23 such integers.
Next, we consider the positive integers $n \leq 100$ that can be expressed as the sum of exactly 5 consecutive positive integers.
The smallest such integer is $1 + 2 + 3 + 4 + 5 = 15$ and the next is $2 + 3 + 4 + 5 + 6 = 20$.
Using an argument similar to that from above, these integers form an arithemetic sequence with first term 15 and common difference 5.
Since $n \leq 100$, these integers are $15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100$.
When we exclude the integers already listed above (30, 50, 70, 90), we obtain

$$15, 20, 25, 35, 40, 45, 55, 60, 65, 75, 80, 85, 95, 100$$

There are 14 such integers.
Next, we consider the positive integers $n \leq 100$ that can be expressed as the sum of exactly 6 consecutive positive integers.
These integers form an arithmetic sequence with first term 21 and common difference 6.
Since $n \leq 100$, these integers are $21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99$.
When we exclude the integers already listed above (45, 75), we obtain

$$21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99$$

There are 12 such integers.
Since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105$ and this is the smallest integer that can be expressed as the sum of 14 consecutive positive integers, then no $n \leq 100$ is the sum of 14 or more consecutive positive integers. (Any sum of 15 or more consecutive positive integers will be larger than 105.)
Therefore, if an integer $n \leq 100$ can be expressed as the sum of $s \geq 4$ consecutive integers, then $s \leq 13$.
We make a table to enumerate the $n \leq 100$ that come from values of s with $7 \leq s \leq 13$ that we have not yet counted:

s	Smallest n	Possible $n \leq 100$	New n
7	28	$28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98$	$28, 49, 56, 77, 84, 91$
8	36	$36, 44, 52, 60, 68, 76, 84, 92, 100$	$36, 44, 52, 68, 76, 92$
9	45	$45, 54, 63, 72, 81, 90, 99$	72
10	55	$55, 65, 75, 85, 95$	None
11	66	$66, 77, 88, 99$	88
12	78	$78, 90$	None
13	91	91	None

In total, there are $23 + 14 + 12 + 6 + 6 + 1 + 1 = 63$ such n.
What do you notice about the n that cannot expressed in this way?

ANSWER: (B)

24. A quadratic equation has two distinct real solutions exactly when its discriminant is positive. For the quadratic equation $x^2 - (r+7)x + r + 87 = 0$, the discriminant is

$$\Delta = (r+7)^2 - 4(1)(r+87) = r^2 + 14r + 49 - 4r - 348 = r^2 + 10r - 299$$

Since $\Delta = r^2 + 10r - 299 = (r+23)(r-13)$ which has roots $r = -23$ and $r = 13$, then $\Delta > 0$ exactly when $r > 13$ or $r < -23$. (To see this, we could picture the parabola with equation $y = x^2 + 10x - 299 = (x+23)(x-13)$ and see where it lies above the x-axis.)

We also want both of the solutions of the original quadratic equation to be negative.
If $r > 13$, then the equation $x^2 - (r+7)x + r + 87 = 0$ is of the form $x^2 - bx + c = 0$ with each of b and c positive.
In this case, if $x < 0$, then $x^2 > 0$ and $-bx > 0$ and $c > 0$ and so $x^2 - bx + c > 0$.
This means that, if $r > 13$, there cannot be negative solutions.
Thus, it must be the case that $r < -23$. This does not guarantee negative solutions, but is a necessary condition.
So we consider $x^2 - (r+7)x + r + 87 = 0$ along with the condition $r < -23$.
This quadratic is of the form $x^2 - bx + c = 0$ with $b < 0$. We do not yet know whether c is positive, negative or zero.
We know that this equation has two distinct real solutions.
Suppose that the quadratic equation $x^2 - bx + c = 0$ has real solutions s and t.
This means that the factors of $x^2 - bx + c$ are $x - s$ and $x - t$.
In other words, $(x-s)(x-t) = x^2 - bx + c$.
Now,

$$(x-s)(x-t) = x^2 - tx - sx + st = x^2 - (s+t)x + st$$

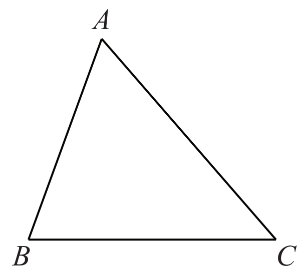
Since $(x-s)(x-t) = x^2 - bx + c$, then $x^2 - (s+t)x + st = x^2 - bx + c$ for all values of x, which means that $b = (s+t)$ and $c = st$.
Since $b < 0$, then it cannot be the case that s and t are both positive, since $b = s + t$.
If $c = 0$, then it must be the case that $s = 0$ or $t = 0$.
If $c < 0$, then it must be the case that one of s and t is positive and the other is negative.
If $c = st$ is positive, then s and t are both positive or both negative, but since $b < 0$, then s and t cannot both be positive, hence are both negative.
Knowing that the equation $x^2 - bx + c = 0$ has two distinct real roots and that $b < 0$, the condition that the two roots are negative is equivalent to the condition that $c > 0$.
Here, $c = r + 87$ and so $c > 0$ exactly when $r > -87$.
Finally, this means that the equation $x^2 - (r+7)x + r + 87 = 0$ has two distinct real roots which are both negative exactly when $-87 < r < -23$.
This means that $p = -87$ and $q = -23$ and so $p^2 + q^2 = 8098$.

ANSWER: (E)

25. In this solution, we will use two geometric results:

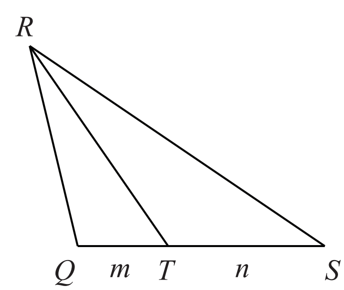
(i) The *Triangle Inequality*
This result says that, in $\triangle ABC$, each of the following inequalities is true:

$$AB + BC > AC \qquad AC + BC > AB \qquad AB + AC > BC$$



This result comes from the fact that the shortest distance between two points is the length of the straight line segment joining those two points.
For example, the shortest distance between the points A and C is the length of the line segment AC. Thus, the path from A to C through a point B not on AC, which has length $AB + BC$, is longer. This explanation tells us that $AB + BC > AC$.

(ii) The *Angle Bisector Theorem*
In the given triangle, we are told that $\angle QRT = \angle SRT$. This tells us that RT is an angle bisector of $\angle QRS$. The Angle Bisector Theorem says that, since RT is the angle bisector of $\angle QRS$, then $\dfrac{QT}{TS} = \dfrac{RQ}{RS}$.



The Angle Bisector Theorem can be proven using the sine law:

In $\triangle RQT$, we have $\dfrac{RQ}{\sin(\angle RTQ)} = \dfrac{QT}{\sin(\angle QRT)}$.

In $\triangle RST$, we have $\dfrac{RS}{\sin(\angle RTS)} = \dfrac{TS}{\sin(\angle SRT)}$.

Dividing the first equation by the second, we obtain

$$\frac{RQ\sin(\angle RTS)}{RS\sin(\angle RTQ)} = \frac{QT\sin(\angle SRT)}{TS\sin(\angle QRT)}$$

Since $\angle QRT = \angle SRT$, then $\sin(\angle QRT) = \sin(\angle SRT)$.
Since $\angle RTQ = 180^\circ - \angle RTS$, then $\sin(\angle RTQ) = \sin(\angle RTS)$.

Combining these three equalities, we obtain $\dfrac{RQ}{RS} = \dfrac{QT}{TS}$, as required.

We now begin our solution to the problem.

By the Angle Bisector Theorem, $\dfrac{RQ}{RS} = \dfrac{QT}{TS} = \dfrac{m}{n}$.

Therefore, we can set $RQ = km$ and $RS = kn$ for some real number $k > 0$.
By the Triangle Inequality, $RQ + RS > QS$.
This is equivalent to the inequality $km + kn > m + n$ or $k(m + n) > m + n$.
Since $m + n > 0$, this is equivalent to $k > 1$.
Using the Triangle Inequality a second time, we know that $RQ + QS > RS$.
This is equivalent to $km + m + n > kn$, which gives $k(n - m) < n + m$.
Since $n > m$, then $n - m > 0$ and so we obtain $k < \dfrac{n+m}{n-m}$.
(Since we already know that $RS > RQ$, a third application of the Triangle Inequality will not give any further information. Can you see why?)

The perimeter, p, of $\triangle QRS$ is $RQ + RS + QS = km + kn + m + n = (k + 1)(m + n)$.
Since $k > 1$, then $p > 2(m + n)$.
Since $2(m + n)$ is an integer, then the smallest possible integer value of p is $2m + 2n + 1$.
Since $k < \dfrac{n+m}{n-m}$, then $p < \left(\dfrac{n+m}{n-m} + 1\right)(n + m)$.
Since $n+m$ is a multiple of $n-m$, then $\left(\dfrac{n+m}{n-m} + 1\right)(n+m)$ is an integer, and so the largest possible integer value of p is $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$.
Every possible value of p between $2m + 2n + 1$ and $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$, inclusive, can actually be achieved. We can see this by starting with point R almost at point T and then continously pulling R away from QS while keeping the ratio $\dfrac{RQ}{RS}$ fixed until the triangle is almost flat with RS along RQ and QS.
We know that the smallest possible integer value of p is $2m + 2n + 1$ and the largest possible integer value of p is $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$.
The number of integers in this range is

$$\left(\left(\frac{n+m}{n-m} + 1\right)(n+m) - 1\right) - (2m + 2n + 1) + 1$$

From the given information, the number of possible integer values of p is $m^2 + 2m - 1$.
Therefore, we obtain the following equivalent equations:

$$\begin{aligned}
\left(\left(\frac{n+m}{n-m} + 1\right)(n+m) - 1\right) - (2m + 2n + 1) + 1 &= m^2 + 2m - 1 \\
\left(\left(\frac{n+m}{n-m} + 1\right)(n+m)\right) - (2m + 2n) &= m^2 + 2m \\
\left(\left(\frac{n+m}{n-m} + \frac{n-m}{n-m}\right)(n+m)\right) - (2m + 2n) &= m^2 + 2m \\
\left(\frac{2n}{n-m}\right)(n+m) - 2m - 2n &= m^2 + 2m \\
\frac{2n^2 + 2nm}{n-m} - 2m - 2n &= m^2 + 2m
\end{aligned}$$

$$
\begin{aligned}
\frac{2n^2+2nm}{n-m} - \frac{2(n+m)(n-m)}{n-m} &= m^2+2m \\
\frac{2n^2+2nm}{n-m} - \frac{2n^2-2m^2}{n-m} &= m^2+2m \\
\frac{2m^2+2nm}{n-m} &= m^2+2m \\
\frac{2m+2n}{n-m} &= m+2 \qquad (\text{since } m \neq 0) \\
2m+2n &= (m+2)(n-m) \\
2m+2n &= nm+2n-m^2-2m \\
0 &= nm-m^2-4m \\
0 &= m(n-m-4)
\end{aligned}
$$

Since $m > 0$, then $n-m-4=0$ and so $n-m=4$.

For an example of such a triangle, suppose that $m=2$ and $n=6$.

Here, $\dfrac{n+m}{n-m} = 2$ and so the minimum possible perimeter is $2n+2m+1=17$ and the maximum possible perimeter is $\left(\dfrac{n+m}{n-m}+1\right)(n+m)-1=23$.

The number of integers between 17 and 23, inclusive, is 7, which equals m^2+2m-1 or $2^2+2(2)-1$, as expected.

Answer: (A)