The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2019 Euclid Contest

Wednesday, April 3, 2019
(in North America and South America)

Thursday, April 4, 2019
(outside of North America and South America)

Solutions

1. (a) *Solution 1*

Since $\frac{3}{4}$ of a jar has a volume of 300 mL, then $\frac{1}{4}$ of a jar has a volume of (300 mL) $\div$ 3 or 100 mL.

Solution 2

Since $\frac{3}{4}$ of a jar has a volume of 300 mL, then the volume of the entire jar is $\frac{4}{3}$(300 mL) or 400 mL.

In this case, the volume of $\frac{1}{4}$ of the jar is (400 mL) $\div 4 = 100$ mL.

(b) We note that since $\dfrac{24}{a} > 3 > 0$, then a is positive.

Since $3 < \dfrac{24}{a}$ and $a > 0$, then $a < \dfrac{24}{3} = 8$.

Since $\dfrac{24}{a} < 4$ and $a > 0$, then $a > \dfrac{24}{4} = 6$.

Since $6 < a < 8$ and a is an integer, then $a = 7$.

Note that it is indeed true that $3 < \dfrac{24}{7} < 4$.

(c) Since x and x^2 appear in the denominators of the equation, then $x \neq 0$.

Multiplying by x^2 and manipulating, we obtain successively

$$\begin{aligned}
\frac{1}{x^2} - \frac{1}{x} &= 2 \\
1 - x &= 2x^2 \\
0 &= 2x^2 + x - 1 \\
0 &= (2x-1)(x+1)
\end{aligned}$$

and so $x = \dfrac{1}{2}$ or $x = -1$.

Checking in the original equation we obtain,

$$\frac{1}{(1/2)^2} - \frac{1}{1/2} = \frac{1}{1/4} - \frac{1}{1/2} = 4 - 2 = 2$$

and

$$\frac{1}{(-1)^2} - \frac{1}{-1} = \frac{1}{1} + 1 = 2$$

and so the solutions to the equation are $x = \dfrac{1}{2}$ and $x = -1$.

2. (a) Since the radius of the large circle is 2, its area is $\pi \cdot 2^2 = 4\pi$.

Since the radius of each small circle is 1, the area of each small circle is $\pi \cdot 1^2 = \pi$.

Since the two small circles are tangent to each other and to the large circle, then their areas do not overlap and are contained entirely within the large circle.

Since the shaded region consists of the part of the large circle that is outside the two small circles, then the shaded area is $4\pi - \pi - \pi = 2\pi$.

(b) Mo starts at 10:00 a.m. and finishes at 11:00 a.m. and so runs for 1 hour.

Mo runs at 6 km/h, and so runs 6 km in 1 hour.

Thus, Kari also runs 6 km.

Since Kari runs at 8 km/h, then Kari runs for $\dfrac{6 \text{ km}}{8 \text{ km/h}} = \dfrac{3}{4}$ h which is 45 minutes.

Since Kari finishes at 11:00 a.m., then Kari started at 10:15 a.m.

(c) The equation $x + 3y = 7$ can be rearranged to $3y = -x + 7$ and $y = -\frac{1}{3}x + \frac{7}{3}$.
Therefore, the line with this equation has slope $-\frac{1}{3}$.
Since the two lines are parallel and the line with equation $y = mx + b$ has slope m, then $m = -\frac{1}{3}$.
Thus, the equation of the second line can be re-written as $y = -\frac{1}{3}x + b$.
Since $(9, 2)$ lies on this line, then $2 = -\frac{1}{3} \cdot 9 + b$ and so $2 = -3 + b$, which gives $b = 5$.

3. (a) Michelle's list consists of 8 numbers and so its average is

$$\frac{5 + 10 + 15 + 16 + 24 + 28 + 33 + 37}{8} = \frac{168}{8} = 21$$

Daphne's list thus consists of 7 numbers (one fewer than in Michelle's list) with an average of 20 (1 less than that of Michelle).
The sum of 7 numbers whose average is 20 is $7 \cdot 20 = 140$.
Since the sum of Michelle's numbers was 168, then Daphne removed the number equal to $168 - 140$ which is 28.

(b) Since $16 = 2^4$ and $32 = 2^5$, then the given equation is equivalent to the following equations

$$\begin{aligned}(2^4)^{15/x} &= (2^5)^{4/3} \\ 2^{60/x} &= 2^{20/3}\end{aligned}$$

This means that $\dfrac{60}{x} = \dfrac{20}{3} = \dfrac{60}{9}$ and so $x = 9$.

(c) Using exponent laws, the following equations are equivalent:

$$\begin{aligned}\frac{2^{2022} + 2^a}{2^{2019}} &= 72 \\ 2^{2022-2019} + 2^{a-2019} &= 72 \\ 2^3 + 2^{a-2019} &= 72 \\ 8 + 2^{a-2019} &= 72 \\ 2^{a-2019} &= 64 \\ 2^{a-2019} &= 2^6\end{aligned}$$

which means that $a - 2019 = 6$ and so $a = 2025$.

4. (a) *Solution 1*
Since $\triangle CDB$ is right-angled at B, then $\angle DCB = 90^\circ - \angle CDB = 30^\circ$.
This means that $\triangle CDB$ is a 30°-60°-90° triangle.
Using the ratios of side lengths in a 30°-60°-90° triangle, $CD : DB = 2 : 1$.
Since $DB = 10$, then $CD = 20$.
Since $\angle CDB = 60^\circ$, then $\angle ADC = 180^\circ - \angle CDB = 120^\circ$.
Since the angles in $\triangle ADC$ add to 180°, then $\angle DAC = 180^\circ - \angle ADC - \angle ACD = 30^\circ$.
This means that $\triangle ADC$ is isosceles with $AD = CD$.
Therefore, $AD = CD = 20$.

Solution 2
Since $\triangle CDB$ is right-angled at B, then $\angle DCB = 90^\circ - \angle CDB = 30^\circ$.
Since $\triangle ACB$ is right-angled at B, then $\angle CAB = 90^\circ - \angle ACB = 90^\circ - (30^\circ + 30^\circ) = 30^\circ$.
This means that each of $\triangle CDB$ and $\triangle ACB$ is a 30°-60°-90° triangle.
Using the ratios of side lengths in a 30°-60°-90° triangle, $CB : DB = \sqrt{3} : 1$.
Since $DB = 10$, then $CB = 10\sqrt{3}$.
Similarly, $AB : CB = \sqrt{3} : 1$.
Since $CB = 10\sqrt{3}$, then $AB = \sqrt{3} \cdot 10\sqrt{3} = 30$.
Finally, this means that $AD = AB - DB = 30 - 10 = 20$.

(b) Since the points $A(d, -d)$ and $B(-d + 12, 2d - 6)$ lie on the same circle centered at the origin, O, then $OA = OB$.
Since distances are always non-negative, the following equations are equivalent:

$$\begin{aligned}
\sqrt{(d-0)^2 + (-d-0)^2} &= \sqrt{((-d+12)-0)^2 + ((2d-6)-0)^2} \\
d^2 + (-d)^2 &= (-d+12)^2 + (2d-6)^2 \\
d^2 + d^2 &= d^2 - 24d + 144 + 4d^2 - 24d + 36 \\
2d^2 &= 5d^2 - 48d + 180 \\
0 &= 3d^2 - 48d + 180 \\
0 &= d^2 - 16d + 60 \\
0 &= (d-10)(d-6)
\end{aligned}$$

and so $d = 10$ or $d = 6$.
We can check that the points $A(10, -10)$ and $B(2, 14)$ are both of distance $\sqrt{200}$ from the origin and the points $A(6, -6)$ and $B(6, 6)$ are both of distance $\sqrt{72}$ from the origin.

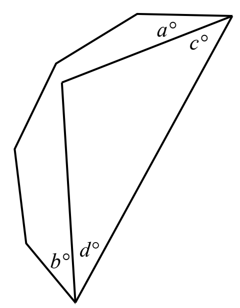
5. (a) First, we note that $\sqrt{50} = 5\sqrt{2}$.
Next, we note that $\sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$ and $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.
From the first of these, we obtain $\sqrt{2} + \sqrt{32} = \sqrt{50}$.
From the second of these, we obtain $\sqrt{8} + \sqrt{18} = \sqrt{50}$.
Thus, $(a, b) = (2, 32)$ and $(a, b) = (8, 18)$ are solutions to the original equation.
(We are not asked to justify why these are the only two solutions.)

(b) From the second equation, we note that $d \neq 0$.
Rearranging this second equation, we obtain $c = kd$.
Substituting into the first equation, we obtain $kd + d = 2000$ or $(k + 1)d = 2000$.
Since $k \geq 0$, note that $k + 1 \geq 1$.
This means that if (c, d) is a solution, then $k + 1$ is a divisor of 2000.
Also, if $k + 1$ is a divisor of 2000, then the equation $(k + 1)d = 2000$ gives us an integer value of d (which is non-zero) from which we can find an integer value of c using the first equation.
Therefore, the values of k that we want to count correspond to the positive divisors of 2000.
Since $2000 = 10 \cdot 10 \cdot 20 = 2^4 \cdot 5^3$, then 2000 has $(4 + 1)(3 + 1) = 20$ positive divisors.
This comes from the fact that if p and q are distinct prime numbers then the positive integer $p^a \cdot q^b$ has $(a + 1)(b + 1)$ positive divisors.
We could list these divisors as

$$1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500, 1000, 2000$$

if we did not know the earlier formula.
Since 2000 has 20 positive divisors, then there are 20 values of k for which the system of equations has at least one integer solution.
For example, if $k+1=8$, then $k=7$. This gives the system $c+d=2000$ and $\frac{c}{d}=7$ which has solution $(c,d)=(1750,250)$.

6. (a) *Solution 1*
The angles in a polygon with n sides have a sum of $(n-2)\cdot 180^\circ$.
This means that the angles in a pentagon have a sum of $3\cdot 180^\circ$ or 540°, which means that each interior angle in a regular pentagon equals $\frac{1}{5}\cdot 540^\circ$ or 108°.
Also, each interior angle in a regular polygon with n sides equals $\frac{n-2}{n}\cdot 180^\circ$. (This is the general version of the statement in the previous sentence.)
Consider the portion of the regular polygon with n sides that lies outside the pentagon and join the points from which the angles that measure a° and b° emanate to form a hexagon.



This polygon has 6 sides, and so the sum of its 6 angles is $4\cdot 180^\circ$.
Four of its angles are the original angles from the n-sided polygon, so each equals $\frac{n-2}{n}\cdot 180^\circ$.
The remaining two angles have measures $a^\circ+c^\circ$ and $b^\circ+d^\circ$.
We are told that $a^\circ+b^\circ=88^\circ$.
Also, the angles that measure c° and d° are two angles in a triangle whose third angle is 108°.
Thus, $c^\circ+d^\circ=180^\circ-108^\circ=72^\circ$.
Therefore,

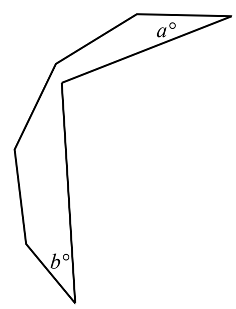
$$\begin{aligned}
4\cdot\frac{n-2}{n}\cdot 180^\circ+88^\circ+72^\circ &= 4\cdot 180^\circ\\
160^\circ &= \left(4-\frac{4(n-2)}{n}\right)\cdot 180^\circ\\
160^\circ &= \frac{4n-(4n-8)}{n}\cdot 180^\circ\\
\frac{160^\circ}{180^\circ} &= \frac{8}{n}\\
\frac{8}{9} &= \frac{8}{n}
\end{aligned}$$

and so the value of n is 9.

Solution 2

The angles in a polygon with n sides have a sum of $(n-2) \cdot 180^\circ$.
This means that the angles in a pentagon have a sum of $3 \cdot 180^\circ$ or 540°, which means that each interior angle in a regular pentagon equals $\frac{1}{5} \cdot 540^\circ$ or 108°.

Also, each interior angle in a regular polygon with n sides equals $\frac{n-2}{n} \cdot 180^\circ$. (This is the general version of the statement in the previous sentence.)
Consider the portion of the regular polygon with n sides that lies outside the pentagon.

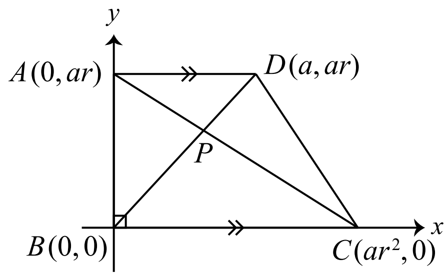


This polygon has 7 sides, and so the sum of its 7 angles is $5 \cdot 180^\circ$.
Four of its angles are the original angles from the n-sided polygon, so each equals $\frac{n-2}{n} \cdot 180^\circ$.
Two of its angles are the angles equal to a° and b°, whose sum is 88°.
Its seventh angle is the reflex angle corresponding to the pentagon's angle of 108°, which equals $360^\circ - 108^\circ$ or 252°.
Therefore,

$$\begin{aligned} 4 \cdot \frac{n-2}{n} \cdot 180^\circ + 88^\circ + 252^\circ &= 5 \cdot 180^\circ \\ 340^\circ &= \left(5 - \frac{4(n-2)}{n}\right) \cdot 180^\circ \\ 340^\circ &= \frac{5n - (4n-8)}{n} \cdot 180^\circ \\ \frac{340^\circ}{180^\circ} &= \frac{n+8}{n} \\ \frac{17}{9} &= \frac{n+8}{n} \\ 17n &= 9(n+8) \\ 17n &= 9n + 72 \\ 8n &= 72 \end{aligned}$$

and so the value of n is 9.

(b) Since the lengths of AD, AB and BC form a geometric sequence, we suppose that these lengths are a, ar and ar^2, respectively, for some real numbers $a > 0$ and $r > 0$.
Since the angles at A and B are both right angles, we assign coordinates to the diagram, putting B at the origin $(0, 0)$, C on the positive x-axis at $(ar^2, 0)$, A on the positive y-axis at $(0, ar)$, and D at (a, ar).



Therefore, the slope of the line segment joining $B(0, 0)$ and $D(a, ar)$ is $\dfrac{ar - 0}{a - 0} = r$.
Also, the slope of the line segment joining $A(0, ar)$ and $C(ar^2, 0)$ is $\dfrac{ar - 0}{0 - ar^2} = -\dfrac{1}{r}$.
Since the product of the slopes of these two line segments is -1, then the segments are perpendicular, as required.

7. (a) Using logarithm and exponent laws, we obtain the following equivalent equations:

$$\begin{aligned}
2\log_2(x-1) &= 1 - \log_2(x+2) \\
2\log_2(x-1) + \log_2(x+2) &= 1 \\
\log_2\left((x-1)^2\right) + \log_2(x+2) &= 1 \\
\log_2\left((x-1)^2(x+2)\right) &= 1 \\
(x-1)^2(x+2) &= 2^1 \\
(x^2 - 2x + 1)(x+2) &= 2 \\
x^3 - 3x + 2 &= 2 \\
x^3 - 3x &= 0 \\
x(x^2 - 3) &= 0
\end{aligned}$$

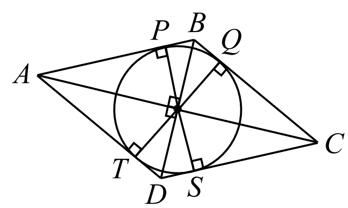
and so $x = 0$ or $x = \sqrt{3}$ or $x = -\sqrt{3}$.
Note that if $x = 0$, then $x - 1 = -1 < 0$ and so $\log_2(x - 1)$ is not defined. Thus, $x \neq 0$.
Note that if $x = -\sqrt{3}$, then $x - 1 = -\sqrt{3} - 1 < 0$ and so $\log_2(x - 1)$ is not defined. Thus, $x \neq -\sqrt{3}$.
If $x = \sqrt{3}$, we can verify that both logarithms in the original equation are defined and that the original equation is true. We could convince ourselves of this with a calculator or we could algebraically verify that raising 2 to the power of both sides gives the same number, so the expressions must actually be equal.
Therefore, $x = \sqrt{3}$ is the only solution.

(b) Let $a = f(f(x))$.
Thus, the equation $f(f(f(x))) = 3$ is equivalent to $f(a) = 3$.
Since $f(a) = a^2 - 2a$, then we obtain the equation $a^2 - 2a = 3$ which gives $a^2 - 2a - 3 = 0$ and $(a - 3)(a + 1) = 0$.
Thus, $a = 3$ or $a = -1$ which means that $f(f(x)) = 3$ or $f(f(x)) = -1$.
Let $b = f(x)$.
Thus, the equations $f(f(x)) = 3$ and $f(f(x)) = -1$ become $f(b) = 3$ and $f(b) = -1$.
If $f(b) = 3$, then $b = f(x) = 3$ or $b = f(x) = -1$ using similar reasoning to above when $f(a) = 3$.
If $f(b) = -1$, then $b^2 - 2b = -1$ and so $b^2 - 2b + 1 = 0$ or $(b - 1)^2 = 0$ which means that $b = f(x) = 1$.
Thus, $f(x) = 3$ or $f(x) = -1$ or $f(x) = 1$.
If $f(x) = 3$, then $x = 3$ or $x = -1$ as above.
If $f(x) = -1$, then $x = 1$ as above.
If $f(x) = 1$, then $x^2 - 2x = 1$ and so $x^2 - 2x - 1 = 0$.
By the quadratic formula,

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Therefore, the solutions to the equation $f(f(f(x))) = 3$ are $x = 3, 1, -1, 1 + \sqrt{2}, 1 - \sqrt{2}$.

8. (a) Since $\angle AOB = \angle BOC = \angle COD = \angle DOA$ and these angles form a complete circle around O, then $\angle AOB = \angle BOC = \angle COD = \angle DOA = \frac{1}{4} \cdot 360^\circ = 90^\circ$.
Join point O to P, B, Q, C, S, D, T, and A.



Since P, Q, S, and T are points of tangency, then the radii meet the sides of $ABCD$ at right angles at these points.
Since $AO = 3$ and $OT = 1$ and $\angle OTA = 90^\circ$, then by the Pythagorean Theorem, $AT = \sqrt{AO^2 - OT^2} = \sqrt{8} = 2\sqrt{2}$.
Since $\triangle OTA$ is right-angled at T, then $\angle TAO + \angle AOT = 90^\circ$.
Since $\angle DOA = 90^\circ$, then $\angle AOT + \angle DOT = 90^\circ$.
Thus, $\angle TAO = \angle DOT$.
This means that $\triangle ATO$ is similar to $\triangle OTD$.
Thus, $\dfrac{DT}{OT} = \dfrac{OT}{AT}$ and so $DT = \dfrac{OT^2}{AT} = \dfrac{1}{2\sqrt{2}}$.
Since DS and DT are tangents to the circle from the same point, then $DS = DT = \dfrac{1}{2\sqrt{2}}$.

(b) Since $0 < x < \frac{\pi}{2}$, then $0 < \cos x < 1$ and $0 < \sin x < 1$.
This means that $0 < \frac{3}{2}\cos x < \frac{3}{2}$ and $0 < \frac{3}{2}\sin x < \frac{3}{2}$. Since $3 < \pi$, then $0 < \frac{3}{2}\cos x < \frac{\pi}{2}$ and $0 < \frac{3}{2}\sin x < \frac{\pi}{2}$.
If Y and Z are angles with $0 < Y < \frac{\pi}{2}$ and $0 < Z < \frac{\pi}{2}$, then $\cos Y = \sin Z$ exactly when $Y + Z = \frac{\pi}{2}$. To see this, we could picture points R and S on the unit circle corresponding to the angles Y and Z; the x-coordinate of R is equal to the y-coordinate of S exactly when the angles Y and Z are complementary.
Therefore, the following equations are equivalent:

$$\begin{aligned}
\cos\left(\frac{3}{2}\cos x\right) &= \sin\left(\frac{3}{2}\sin x\right) \\
\frac{3}{2}\cos x + \frac{3}{2}\sin x &= \frac{\pi}{2} \\
\cos x + \sin x &= \frac{\pi}{3} \\
(\sin x + \cos x)^2 &= \frac{\pi^2}{9} \\
\sin^2 x + 2\sin x\cos x + \cos^2 x &= \frac{\pi^2}{9} \\
2\sin x\cos x + (\sin^2 x + \cos^2 x) &= \frac{\pi^2}{9} \\
\sin 2x + 1 &= \frac{\pi^2}{9} \\
\sin 2x &= \frac{\pi^2 - 9}{9}
\end{aligned}$$

Therefore, the only possible value of $\sin 2x$ is $\frac{\pi^2 - 9}{9}$.

9. (a) By definition, $f(2,5) = \frac{2}{5} + \frac{5}{2} + \frac{1}{2 \cdot 5} = \frac{2 \cdot 2 + 5 \cdot 5 + 1}{2 \cdot 5} = \frac{4 + 25 + 1}{10} = \frac{30}{10} = 3$.

(b) By definition, $f(a,a) = \frac{a}{a} + \frac{a}{a} + \frac{1}{a^2} = 2 + \frac{1}{a^2}$.
For $2 + \frac{1}{a^2}$ to be an integer, it must be the case that $\frac{1}{a^2}$ is an integer.
For $\frac{1}{a^2}$ to be an integer and since a^2 is an integer, a^2 needs to be a divisor of 1.
Since a^2 is positive, then $a^2 = 1$.
Since a is a positive integer, then $a = 1$.
Thus, the only positive integer a for which $f(a,a)$ is an integer is $a = 1$.

(c) Suppose that a and b are positive integers for which $f(a,b)$ is an integer.
Assume that $k = f(a,b)$ is not a multiple of 3.
We will show that there must be a contradiction, which will lead to the conclusion that k must be a multiple of 3.
By definition, $k = f(a,b) = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}$.
Multiplying by ab, we obtain $kab = a^2+b^2+1$, which we re-write as $a^2-(kb)a+(b^2+1) = 0$.
We treat this as a quadratic equation in a with coefficients in terms of the variables b and k.
Solving for a in terms of b and k using the quadratic formula, we obtain

$$a = \frac{kb \pm \sqrt{(-kb)^2 - 4(1)(b^2+1)}}{2} = \frac{kb \pm \sqrt{k^2b^2 - 4b^2 - 4}}{2}$$

Since a is an integer, then the discriminant $k^2b^2 - 4b^2 - 4$ must be a perfect square.
Re-writing the discriminant, we obtain

$$k^2b^2 - 4b^2 - 4 = b^2(k^2-4) - 4 = b^2(k-2)(k+2) - 4$$

Since k is not a multiple of 3, then it is either 1 more than a multiple of 3 or it is 2 more than a multiple of 3.
If k is 1 more than a multiple of 3, then $k+2$ is a multiple of 3.
If k is 2 more than a multiple of 3, then $k-2$ is a multiple of 3.
In either case, $(k-2)(k+2)$ is a multiple of 3, say $(k-2)(k+2) = 3m$ for some integer m.
This means that the discriminant can be re-written again as

$$b^2(3m) - 4 = 3(b^2m - 2) + 2$$

In other words, the discriminant is itself 2 more than a multiple of 3.
However, every perfect square is either a multiple of 3 or one more than a multiple of 3:

> Suppose that r is an integer and consider r^2.
> The integer r can be written as one of $3q$, $3q+1$, $3q+2$, for some integer q.
> These three cases give
>
> $$\begin{aligned} (3q)^2 &= 9q^2 = 3(3q^2) \\ (3q+1)^2 &= 9q^2 + 6q + 1 = 3(3q^2+2q) + 1 \\ (3q+2)^2 &= 9q^2 + 12q + 4 = 3(3q^2+4q+1) + 1 \end{aligned}$$
>
> and so r^2 is either a multiple of 3 or 1 more than a multiple of 3.

We have determined that the discriminant is a perfect square and is 2 more than a multiple of 3. This is a contradiction.
This means that our initial assumption must be incorrect, and so $k = f(a, b)$ must be a multiple of 3.

(d) *Solution 1*
We find additional pairs of positive integers (a, b) with $f(a, b) = 3$.
Suppose that $f(a, b) = 3$.
This is equivalent to the equations $\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} = 3$ and $a^2 + b^2 - 3ab + 1 = 0$.
Then

$$\begin{aligned} f(b, 3b-a) - 3 &= \frac{b}{3b-a} + \frac{3b-a}{b} + \frac{1}{b(3b-a)} - 3 \\ &= \frac{b^2 + (3b-a)^2 + 1 - 3b(3b-a)}{b(3b-a)} \\ &= \frac{b^2 + (3b-a)(3b-a) + 1 - 3b(3b-a)}{b(3b-a)} \\ &= \frac{b^2 - a(3b-a) + 1}{b(3b-a)} \\ &= \frac{b^2 + a^2 - 3ab + 1}{b(3b-a)} \\ &= 0 \end{aligned}$$

Therefore, if $f(a, b) = 3$, then $f(b, 3b - a) = 3$.
The equation $f(1, 2) = 3$ gives $f(2, 3(2) - 1) = f(2, 5) = 3$.
The equation $f(2, 5) = 3$ gives $f(5, 3(5) - 2) = f(5, 13) = 3$.
The equation $f(5, 13) = 3$ gives $f(13, 3(13) - 5) = f(13, 34) = 3$.
The equation $f(13, 34) = 3$ gives $f(34, 3(34) - 13) = f(34, 89) = 3$.
The equation $f(34, 89) = 3$ gives $f(89, 3(89) - 34) = f(89, 233) = 3$.
Thus, the pairs $(a, b) = (5, 13), (13, 34), (34, 89), (89, 233)$ satisfy the requirements.

Solution 2
From (a), we know that $f(2, 5) = 3$.
Since the function $f(a, b)$ is symmetric in a and b (that is, a and b can be switched without changing the value of the function), then $f(5, 2) = 3$.
Consider the equation $f(5, b) = 3$. We know that $b = 2$ is a solution, but is there another solution?
By definition, $f(5, b) = \frac{5}{b} + \frac{b}{5} + \frac{1}{5b}$.
Thus, $f(5, b) = 3$ gives the following equivalent equations:

$$\begin{aligned} \frac{5}{b} + \frac{b}{5} + \frac{1}{5b} &= 3 \\ 25 + b^2 + 1 &= 15b \\ b^2 - 15b + 26 &= 0 \\ (b-2)(b-13) &= 0 \end{aligned}$$

and so $b = 2$ or $b = 13$. This means that $f(5, 13) = 3$ and so $(a, b) = (5, 13)$ has the property that $f(a, b)$ is an integer.

From $f(5,13)=3$, we get $f(13,5)=3$.
As above, we consider the equation $f(13,b)=3$, for which $b=5$ is a solution.
We obtain the equivalent equations

$$\begin{aligned}\frac{13}{b}+\frac{b}{13}+\frac{1}{13b}&=3\\ 169+b^2+1&=39b\\ b^2-39b+170&=0\\ (b-5)(b-34)&=0\end{aligned}$$

and so $b=5$ or $b=34$. This means that $f(13,34)=3$ and so $(a,b)=(13,34)$ has the property that $f(a,b)$ is an integer.
Continuing in a similar manner, we can also find that $f(34,89)$ and $f(89,233)$ are both integers.
Thus, the pairs $(a,b)=(5,13),(13,34),(34,89),(89,233)$ satisfy the requirements.

Solution 3
Note that

$$\begin{aligned}f(5,13)&=\frac{5}{13}+\frac{13}{5}+\frac{1}{5\cdot 13}=\frac{5^2+13^2+1}{65}=\frac{195}{65}=3\\ f(13,34)&=\frac{13}{34}+\frac{34}{13}+\frac{1}{13\cdot 34}=\frac{13^2+34^2+1}{442}=\frac{1326}{442}=3\\ f(34,89)&=\frac{34}{89}+\frac{89}{34}+\frac{1}{34\cdot 89}=\frac{34^2+89^2+1}{3026}=\frac{9078}{3026}=3\\ f(89,233)&=\frac{89}{233}+\frac{233}{89}+\frac{1}{89\cdot 233}=\frac{89^2+233^2+1}{20737}=\frac{62\,211}{20\,737}=3\end{aligned}$$

and so the pairs $(a,b)=(5,13),(13,34),(34,89),(89,233)$ satisfy the requirements.

Where do these pairs come from?
We define the Fibonacci sequence $F_1,F_2,F_3,F_4,\ldots$ by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ when $n\geq 3$.
Thus, the Fibonacci sequence begins $1,1,2,3,5,8,13,21,34,55,89,\ldots$.
The pairs (a,b) found above are of the form (F_{2k-1},F_{2k+1}) for integers $k\geq 3$.
We note that

$$\begin{aligned}f(F_{2k-1},F_{2k+1})&=\frac{F_{2k-1}}{F_{2k+1}}+\frac{F_{2k+1}}{F_{2k-1}}+\frac{1}{F_{2k-1}F_{2k+1}}\\ &=\frac{(F_{2k-1})^2+(F_{2k+1})^2+1}{F_{2k-1}F_{2k+1}}\\ &=\frac{(F_{2k-1})^2+(F_{2k}+F_{2k-1})^2+1}{F_{2k-1}(F_{2k}+F_{2k-1})}\\ &=\frac{2(F_{2k-1})^2+2F_{2k}F_{2k-1}+(F_{2k})^2+1}{(F_{2k-1})^2+F_{2k}F_{2k-1}}\\ &=\frac{2(F_{2k-1})^2+2F_{2k}F_{2k-1}}{(F_{2k-1})^2+F_{2k}F_{2k-1}}+\frac{(F_{2k})^2+1}{(F_{2k-1})^2+F_{2k}F_{2k-1}}\\ &=2+\frac{(F_{2k})^2+1}{F_{2k-1}F_{2k+1}}\end{aligned}$$

This means that $f(F_{2k-1}, F_{2k+1}) = 3$ if and only if $\dfrac{(F_{2k})^2 + 1}{F_{2k-1}F_{2k+1}} = 1$, or equivalently if and only if $(F_{2k})^2 + 1 = F_{2k-1}F_{2k+1}$, or $(F_{2k})^2 - F_{2k-1}F_{2k+1} = -1$.
We note that $(F_2)^2 - F_1F_3 = 1^2 - 1 \cdot 2 = -1$ and $(F_4)^2 - F_3F_5 = 3^2 - 2 \cdot 5 = -1$ so this is true when $k = 1$ and $k = 2$.
Furthermore, we note that

$$\begin{aligned}
(F_{2k+2})^2 - F_{2k+1}F_{2k+3} &= (F_{2k+2})^2 - F_{2k+1}(F_{2k+2} + F_{2k+1}) \\
&= (F_{2k+2})^2 - F_{2k+1}F_{2k+2} - (F_{2k+1})^2 \\
&= F_{2k+2}(F_{2k+2} - F_{2k+1}) - (F_{2k+1})^2 \\
&= F_{2k+2}F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k+1} + F_{2k})F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k})^2 + F_{2k+1}F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k})^2 + F_{2k+1}(F_{2k} - F_{2k+1}) \\
&= (F_{2k})^2 + F_{2k+1}(-F_{2k-1}) \\
&= (F_{2k})^2 - F_{2k+1}F_{2k-1}
\end{aligned}$$

which means that since $(F_4)^2 - F_3F_5 = -1$, then $(F_6)^2 - F_5F_7 = -1$, which means that $(F_8)^2 - F_7F_9 = -1$, and so on.
Continuing in this way, $(F_{2k})^2 - F_{2k-1}F_{2k+1} = -1$ for all positive integers $k \geq 1$, which in turn means that $f(F_{2k-1}, F_{2k+1}) = 3$, as required.

10. (a) On her first two turns, Brigitte either chooses two cards of the same colour or two cards of different colours. If she chooses two cards of different colours, then on her third turn, she must choose a card that matches one of the cards that she already has.
Therefore, the game ends on or before Brigitte's third turn.
Thus, if Amir wins, he wins on his second turn or on his third turn. (He cannot win on his first turn.)

For Amir to win on his second turn, the second card he chooses must match the first card that he chooses.
On this second turn, there will be 5 cards in his hand, of which 1 matches the colour of the first card that he chose.
Therefore, the probability that Amir wins on his second turn is $\frac{1}{5}$.
Note that there is no restriction on the first card that he chooses or the first card that Brigitte chooses.

For Amir to win on his third turn, the following conditions must be true: (i) the colour of the second card that he chooses is different from the colour of the first card that he chooses, (ii) the colour of the second card that Brigitte chooses is different from the colour of the first card that she chooses, and (iii) the colour of the third card that Amir chooses matches the colour of one of the first two cards.
The probability of (i) is $\frac{4}{5}$, since he must choose any card other than the one that matches the first one.
The probability of (ii) is $\frac{2}{3}$, since Brigitte must choose either of the cards that does not match her first card.
The probability of (iii) is $\frac{2}{4}$, since Amir can choose either of the 2 cards that matches one of the first two cards that he chose.
Again, the cards that Amir and Brigitte choose on their first turns do not matter.
Thus, the probability that Amir wins on his third turn is $\frac{4}{5} \cdot \frac{2}{3} \cdot \frac{2}{4}$ which equals $\frac{4}{15}$.

Finally, the probabilty that Amir wins the game is thus $\frac{1}{5} + \frac{4}{15}$ which equals $\frac{7}{15}$.

(b) Suppose that, after flipping the first 13 coins, the probability that there is an even number of heads is p.
Then the probability that there is an odd number of heads is $1 - p$.
When the 14th coin is flipped, the probability of heads is h_{14} and the probability of not heads is $1 - h_{14}$.
After the 14th coin is flipped, there can be an even number of heads if the first 13 include an even number of heads and the 14th is not heads, or if the first 13 include an odd number of heads and the 14th is heads.
The probability of this is $p(1 - h_{14}) + (1 - p)h_{14}$.
Therefore,

$$\begin{aligned} p(1 - h_{14}) + (1 - p)h_{14} &= \tfrac{1}{2} \\ 2p - 2ph_{14} + 2h_{14} - 2ph_{14} &= 1 \\ 0 &= 4ph_{14} - 2p - 2h_{14} + 1 \\ 0 &= 2p(2h_{14} - 1) - (2h14 - 1) \\ 0 &= (2p - 1)(2h_{14} - 1) \end{aligned}$$

Therefore, either $h_{14} = \frac{1}{2}$ or $p = \frac{1}{2}$.
If $h_{14} = \frac{1}{2}$, we have proven the result.
If $h_{14} \neq \frac{1}{2}$, then $p = \frac{1}{2}$.
This would mean that the probability of getting an even number of heads when the first 13 coins are flipped is $\frac{1}{2}$.
We could repeat the argument above to conclude that either $h_{13} = \frac{1}{2}$ or the probability of obtaining an even number of heads when the first 12 coins are flipped is $\frac{1}{2}$.
Continuing in this way, either one of $h_{14}, h_{13}, \ldots, h_3, h_2$ will equal $\frac{1}{2}$, or the probability of obtaining an even number of heads when 1 coin is flipped is $\frac{1}{2}$.
This last statement is equivalent to saying that the probability of obtaining a head with the first coin is $\frac{1}{2}$ (that is, $h_1 = \frac{1}{2}$).
Therefore, at least one of $h_1, h_2, \ldots, h_{13}, h_{14}$ must equal $\frac{1}{2}$.

(c) For the sum of the two digits printed to be 2, each digit must equal 1.
Thus, $S(2) = p_1q_1$.
For the sum of the two digits printed to be 12, each digit must equal 6.
Thus, $S(12) = p_6q_6$.
For the sum of the two digits printed to be 7, the digits must be 1 and 6, or 2 and 5, or 3 and 4, or 4 and 3, or 5 and 2, or 6 and 1.
Thus, $S(7) = p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1$.
Since $S(2) = S(12)$, then $p_1q_1 = p_6q_6$.
Since $S(2) > 0$ and $S(12) > 0$, then $p_1, q_1, p_6, q_6 > 0$.
If $p_1 = p_6$, then we can divide both sides of $p_1q_1 = p_6q_6$ by $p_1 = p_6$ to obtain $q_1 = q_6$.
If $q_1 = q_6$, then we can divide both sides of $p_1q_1 = p_6q_6$ by $q_1 = q_6$ to obtain $p_1 = p_6$.
Therefore, if we can show that either $p_1 = p_6$ or $q_1 = q_6$, our result will be true.
Suppose that $p_1 \neq p_6$ and $q_1 \neq q_6$.
Since $S(2) = \frac{1}{2}S(7)$ and $S(12) = \frac{1}{2}S(7)$, then

$$\begin{aligned}
S(7) - \tfrac{1}{2}S(7) - \tfrac{1}{2}S(7) &= 0 \\
S(7) - S(2) - S(12) &= 0 \\
p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 - p_1q_1 - p_6q_6 &= 0 \\
p_1q_6 + p_6q_1 - p_1q_1 - p_6q_6 + (p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2) &= 0 \\
(p_1 - p_6)(q_6 - q_1) + (p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2) &= 0 \\
p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 &= -(p_1 - p_6)(q_6 - q_1) \\
p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 &= (p_1 - p_6)(q_1 - q_6)
\end{aligned}$$

Since $p_2, p_3, p_4, p_5, q_2, q_3, q_4, q_5 \geq 0$, then $p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 \geq 0$.
From this, $(p_1 - p_6)(q_1 - q_6) \geq 0$.
Since $p_1 \neq p_6$, then either $p_1 > p_6$ or $p_1 < p_6$.
If $p_1 > p_6$, then $p_1 - p_6 > 0$ and so $(p_1 - p_6)(q_1 - q_6) \geq 0$ tells us that $q_1 - q_6 > 0$ which means $q_1 > q_6$.
But we know that $p_1q_1 = p_6q_6$ and $p_1, q_1, p_6, q_6 > 0$ so we cannot have $p_1 > p_6$ and $q_1 > q_6$.
If $p_1 < p_6$, then $p_1 - p_6 < 0$ and so $(p_1 - p_6)(q_1 - q_6) \geq 0$ tells us that $q_1 - q_6 < 0$ which means $q_1 < q_6$.
But we know that $p_1q_1 = p_6q_6$ and $p_1, q_1, p_6, q_6 > 0$ so we cannot have $p_1 < p_6$ and $q_1 < q_6$.
This is a contradiction.
Therefore, since we cannot have $p_1 > p_6$ or $p_1 < p_6$, it must be the case that $p_1 = p_6$ which means that $q_1 = q_6$, as required.