The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 3, 2019
(in North America and South America)

Thursday, April 4, 2019
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions must be supplied by your supervising teacher and inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked (lightbulb), place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

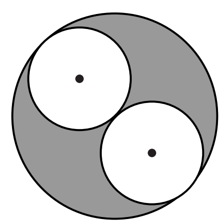
Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) Joyce has two identical jars. The first jar is $\frac{3}{4}$ full of water and contains 300 mL of water. The second jar is $\frac{1}{4}$ full of water. How much water, in mL, does the second jar contain?

   (b) What integer a satisfies $3 < \dfrac{24}{a} < 4$?

   (c) If $\dfrac{1}{x^2} - \dfrac{1}{x} = 2$, determine all possible values of x.

2. (a) In the diagram, two small circles of radius 1 are tangent to each other and to a larger circle of radius 2. What is the area of the shaded region?



   (b) Kari jogs at a constant speed of 8 km/h. Mo jogs at a constant speed of 6 km/h. Kari and Mo jog from the same starting point to the same finishing point along a straight road. Mo starts at 10:00 a.m. Kari and Mo both finish at 11:00 a.m. At what time did Kari start to jog?

   (c) The line with equation $x+3y = 7$ is parallel to the line with equation $y = mx+b$. The line with equation $y = mx + b$ passes through the point $(9, 2)$. Determine the value of b.

3. (a) Michelle calculates the average of the following numbers:

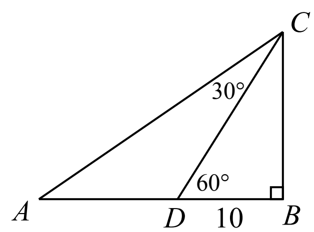
$$5, 10, 15, 16, 24, 28, 33, 37$$

Daphne removes one number and calculates the average of the remaining numbers. The average that Daphne calculates is one less than the average that Michelle calculates. Which number does Daphne remove?

(b) If $16^{\frac{15}{x}} = 32^{\frac{4}{3}}$, what is the value of x?

(c) Suppose that $\dfrac{2^{2022} + 2^a}{2^{2019}} = 72$. Determine the value of a.

4. (a) In the diagram, $\triangle ABC$ has a right angle at B and point D lies on AB. If $DB = 10$, $\angle ACD = 30^\circ$ and $\angle CDB = 60^\circ$, what is the length of AD?



(b) The points $A(d, -d)$ and $B(-d + 12, 2d - 6)$ both lie on a circle centered at the origin. Determine the possible values of d.

5. (a) Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a} + \sqrt{b} = \sqrt{50}$.

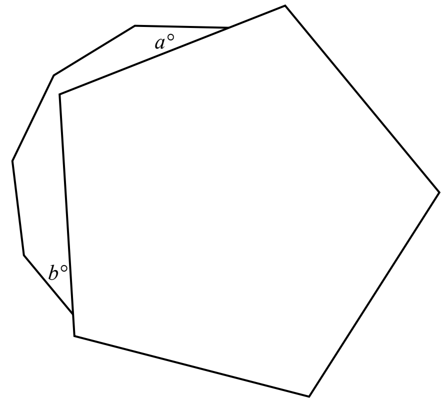
(b) Consider the system of equations:

$$c + d = 2000$$
$$\frac{c}{d} = k$$

Determine the number of integers k with $k \geq 0$ for which there is at least one pair of integers (c, d) that is a solution to the system.

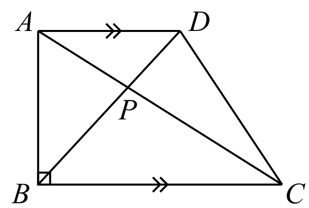
6\. (a) A regular pentagon covers part of another regular polygon, as shown. This regular polygon has n sides, five of which are completely or partially visible. In the diagram, the sum of the measures of the angles marked a° and b° is 88°. Determine the value of n.

(The side lengths of a *regular polygon* are all equal, as are the measures of its interior angles.)



(b) In trapezoid $ABCD$, BC is parallel to AD and BC is perpendicular to AB. Also, the lengths of AD, AB and BC, in that order, form a geometric sequence. Prove that AC is perpendicular to BD.

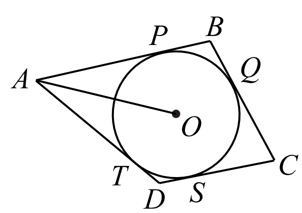
(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant.)



7\. (a) Determine all real numbers x for which $2\log_2(x-1) = 1 - \log_2(x+2)$.

(b) Consider the function $f(x) = x^2 - 2x$. Determine all real numbers x that satisfy the equation $f(f(f(x))) = 3$.

8\. (a) A circle has centre O and radius 1. Quadrilateral $ABCD$ has all 4 sides tangent to the circle at points P, Q, S, and T, as shown. Also, $\angle AOB = \angle BOC = \angle COD = \angle DOA$. If $AO = 3$, determine the length of DS.



(b) Suppose that x satisfies $0 < x < \frac{\pi}{2}$ and $\cos\left(\frac{3}{2}\cos x\right) = \sin\left(\frac{3}{2}\sin x\right)$.

Determine all possible values of $\sin 2x$, expressing your answers in the form $\dfrac{a\pi^2 + b\pi + c}{d}$ where a, b, c, d are integers.

9. For positive integers a and b, define $f(a,b) = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}$.
For example, the value of $f(1,2)$ is 3.

(a) Determine the value of $f(2,5)$.

(b) Determine all positive integers a for which $f(a,a)$ is an integer.

(c) If a and b are positive integers and $f(a,b)$ is an integer, prove that $f(a,b)$ must be a multiple of 3.

(d) Determine four pairs of positive integers (a,b), with $2 < a < b$, for which $f(a,b)$ is an integer.

10. (a) Amir and Brigitte play a card game. Amir starts with a hand of 6 cards: 2 red, 2 yellow and 2 green. Brigitte starts with a hand of 4 cards: 2 purple and 2 white. Amir plays first. Amir and Brigitte alternate turns. On each turn, the current player chooses one of their own cards at random and places it on the table. The cards remain on the table for the rest of the game. A player wins and the game ends when they have placed two cards of the same colour on the table. Determine the probability that Amir wins the game.

(b) Carlos has 14 coins, numbered 1 to 14. Each coin has exactly one face called "heads". When flipped, coins $1, 2, 3, \ldots, 13, 14$ land heads with probabilities $h_1, h_2, h_3, \ldots, h_{13}, h_{14}$, respectively. When Carlos flips each of the 14 coins exactly once, the probability that an even number of coins land heads is exactly $\frac{1}{2}$. Must there be a k between 1 and 14, inclusive, for which $h_k = \frac{1}{2}$? Prove your answer.

(c) Serge and Lis each have a machine that prints a digit from 1 to 6. Serge's machine prints the digits $1, 2, 3, 4, 5, 6$ with probability p_1, p_2, p_3, p_4, p_5, p_6, respectively. Lis's machine prints the digits $1, 2, 3, 4, 5, 6$ with probability q_1, q_2, q_3, q_4, q_5, q_6, respectively. Each of the machines prints one digit. Let $S(i)$ be the probability that the sum of the two digits printed is i. If $S(2) = S(12) = \frac{1}{2}S(7)$ and $S(7) > 0$, prove that $p_1 = p_6$ and $q_1 = q_6$.

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For students...

Thank you for writing the 2019 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2019 Canadian Senior Mathematics Contest, which will be written in November 2019.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2019/2020 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results