The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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# *2019 Cayley Contest*

**(Grade 10)**

**Tuesday, February 26, 2019**
**(in North America and South America)**

**Wednesday, February 27, 2019**
**(outside of North America and South America)**

*Solutions*

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1. Evaluating, $2 \times 0 + 1 - 9 = 0 + 1 - 9 = -8$.

   Answer: (A)

2. Kai was born 25 years before 2020 and so was born in the year $2020 - 25 = 1995$.

   Answer: (C)

3. Since 38% of students received a muffin, then $100\% - 38\% = 62\%$ of students did not receive a muffin.
   Alternatively, using the percentages of students who received yogurt, fruit or a granola bar, we see that $10\% + 27\% + 25\% = 62\%$ did not receive a muffin.

   Answer: (D)

4. Re-arranging the order of the numbers being multiplied,

$$(2 \times \tfrac{1}{3}) \times (3 \times \tfrac{1}{2}) = 2 \times \tfrac{1}{2} \times 3 \times \tfrac{1}{3} = (2 \times \tfrac{1}{2}) \times (3 \times \tfrac{1}{3}) = 1 \times 1 = 1$$

   Answer: (C)

5. Since $10d + 8 = 528$, then $10d = 520$ and so $\dfrac{10d}{5} = \dfrac{520}{5}$ which gives $2d = 104$.

   Answer: (A)

6. The line with equation $y = x + 4$ has a $y$-intercept of 4.
   When the line is translated 6 units downwards, all points on the line are translated 6 units down.
   This moves the $y$-intercept from 4 to $4 - 6 = -2$.

   Answer: (E)

7. Since the average of 2, $x$ and 10 is $x$, then $\dfrac{2 + x + 10}{3} = x$.
   Multiplying by 3, we obtain $2 + x + 10 = 3x$.
   Re-arranging, we obtain $x + 12 = 3x$ and then $2x = 12$ which gives $x = 6$.

   Answer: (E)

8. To get from $P$ to $A$, Alain travels 5 units right and 4 units up, for a total distance of $5 + 4 = 9$ units. (Any path from $P$ to $A$ that only moves right and up will have this same length.)
   To get from $P$ to $B$, Alain travels 6 units right and 2 units up, for a total distance of 8 units.
   To get from $P$ to $C$, Alain travels 3 units right and 3 units up, for a total distance of 6 units.
   To get from $P$ to $D$, Alain travels 5 units right and 1 unit up, for a total distance of 6 units.
   To get from $P$ to $E$, Alain travels 1 unit right and 4 units up, for a total distance of 5 units.
   Therefore, the shortest path is from $P$ to $E$.

   Answer: (E)

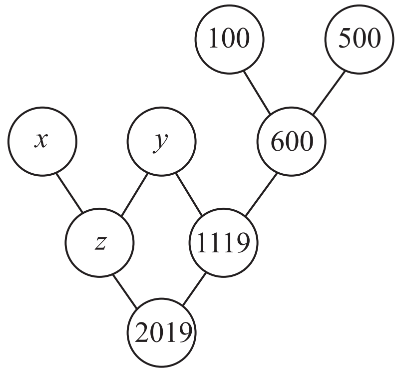
9. Since $(pq)(qr)(rp) = 16$, then $pqqrrp = 16$ or $ppqqrr = 16$ which gives $p^2q^2r^2 = 16$.
   Thus, $(pqr)^2 = 16$ and so $pqr = \pm 4$.
   Using the given answers, $pqr$ is positive and so $pqr = 4$.

   Answer: (C)

10. Matilda and Ellie each take $\frac{1}{2}$ of the wall.
    Matilda paints $\frac{1}{2}$ of her half, or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the entire wall.
    Ellie paints $\frac{1}{3}$ of her half, or $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the entire wall.
    Therefore, $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$ of the wall is painted red.

    Answer: (A)

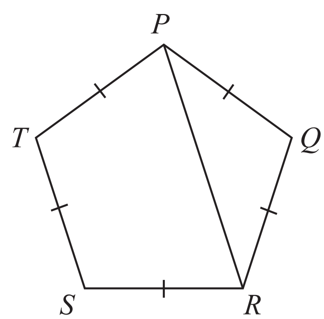
11. We let the values in the two unlabelled circles be $y$ and $z$, as shown.




From the given rules, $y + 600 = 1119$ and so $y = 519$.
Also, $z + 1119 = 2019$ and so $z = 900$.
Finally, $x + y = z$ and so $x = z - y = 900 - 519 = 381$.

Answer: (B)

12. Join $P$ to $R$.




Since $PQRST$ is a regular pentagon, then $\angle PQR = \angle QRS = 108^\circ$.
Since $PQ = QR$, then $\triangle PQR$ is isosceles with $\angle QPR = \angle QRP$.
Since $\angle PQR = 108^\circ$, then

$$\begin{aligned} \angle PQR + \angle QPR + \angle QRP &= 180^\circ \\ 108^\circ + 2\angle QRP &= 180^\circ \\ 2\angle QRP &= 72^\circ \\ \angle QRP &= 36^\circ \end{aligned}$$

Since $\angle QRS = 108^\circ$, then $\angle PRS = \angle QRS - \angle QRP = 108^\circ - 36^\circ = 72^\circ$.

Answer: (A)

13. From the ones column, we see that $3 + 2 + q$ must have a ones digit of 2.
Since $q$ is between 1 and 9, inclusive, then $3 + 2 + q$ is beween 6 and 14.
Since its ones digit is 2, then $3 + 2 + q = 12$ and so $q = 7$.
This also means that there is a carry of 1 into the tens column.
From the tens column, we see that $1 + 6 + p + 8$ must have a ones digit of 4.
Since $p$ is between 1 and 9, inclusive, then $1 + 6 + p + 8$ is beween 16 and 24.
Since its ones digit is 4, then $1 + 6 + p + 8 = 24$ and so $p = 9$.

This also means that there is a carry of 2 into the hundreds column.
From the hundreds column, we see that $2 + n + 7 + 5$ must have a ones digit of 0.
Since $n$ is between 1 and 9, inclusive, then $2 + n + 7 + 5$ is beween 15 and 23.
Since its ones digit is 0, then $2 + n + 7 + 5 = 20$ and so $n = 6$.
This also means that there is a carry of 2 into the thousands column.
This means that $m = 2$.
This gives

$$\begin{array}{rr} & 663 \\ & 792 \\ + & 587 \\ \hline & 2042 \end{array}$$

Thus, we have $m + n + p + q = 2 + 6 + 9 + 7 = 24$.

Answer: (B)

14. Each letter A, B, C, D, E appears exactly once in each column and each row.
The entry in the first column, second row cannot be A or E or B (the entries already present in that column) and cannot be C or A (the entries already present in that row).
Therefore, the entry in the first column, second row must be D.
This means that the entry in the first column, fourth row must be C.
The entry in the fifth column, second row cannot be D or C or A or E and so must be B.
This means that the entry in the second column, second row must be E.
Using similar arguments, the entries in the first row, third and fourth columns must be D and B, respectively.
This means that the entry in the second column, first row must be C.
Using similar arguments, the entries in the fifth row, second column must be A.
Also, the entry in the third row, second column must be D.
This means that the letter that goes in the square marked with $*$ must be B.
We can complete the grid as follows:

| | | | | |
|---|---|---|---|---|
| A | C | D | B | E |
| D | E | C | A | B |
| E | D | B | C | A |
| C | B | A | E | D |
| B | A | E | D | C |

Answer: (B)

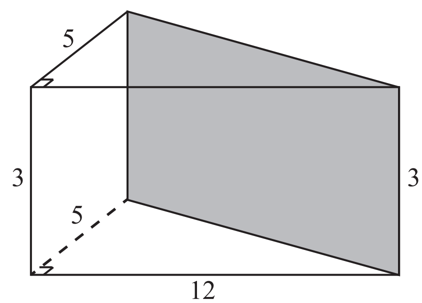
15. The slope of line segment $PR$ is $\dfrac{2-1}{0-4}$ which equals $-\dfrac{1}{4}$.
Since $\angle QPR = 90^\circ$, then $PQ$ and $PR$ are perpendicular.
This means that the slopes of $PQ$ and $PR$ have a product of $-1$.
Since the slope of $PR$ is $-\dfrac{1}{4}$, then the slope of $PQ$ is 4.
Since the "run" of $PQ$ is $2 - 0 = 2$, then the "rise" of $PQ$ must be $4 \times 2 = 8$.
Thus, $s - 2 = 8$ and so $s = 10$.

Answer: (C)

16. Suppose that there are $p$ people behind Kaukab.
This means that there are $2p$ people ahead of her.
Including Kaukab, the total number of people in line is $n = p + 2p + 1 = 3p + 1$, which is one more than a multiple of 3.
Of the given choices (23, 20, 24, 21, 25), the only one that is one more than a multiple of 3 is 25, which equals $3 \times 8 + 1$.
Therefore, a possible value for $n$ is 25.

ANSWER: (E)

17. Consider the triangular-based prism on the front of the rectangular prism.
This prism has five faces: a rectangle on the front, a rectangle on the left, a triangle on the bottom, a triangle on the top, and a rectangle on the back.




The rectangle on the front measures $3 \times 12$ and so has area 36.
The rectangle on the left measures $3 \times 5$ and so has area 15.
The triangles on the top and bottom each are right-angled and have legs of length 5 and 12. This means that each has area $\frac{1}{2} \times 12 \times 5 = 30$.
The rectangle on the back has height 3. The length of this rectangle is the length of the diagonal of the bottom face of the rectangular prism. By the Pythagorean Theorem, this length is $\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$. Thus, this rectangle is $3 \times 13$ and so has area 39.
In total, the surface area of the triangular prism is thus $36 + 15 + 2 \times 30 + 39 = 150$.

ANSWER: (D)

18. André runs for 10 seconds at a speed of $y$ m/s.
Therefore, André runs $10y$ m.
Carl runs for 20 seconds before André starts to run and then 10 seconds while André is running. Thus, Carl runs for 30 seconds.
Since Carl runs at a speed of $x$ m/s, then Carl runs $30x$ m.
Since André and Carl run the same distance, then $30x$ m $= 10y$ m, which means that $\dfrac{y}{x} = 3$.
Thus, $y : x = 3 : 1$.

ANSWER: (D)

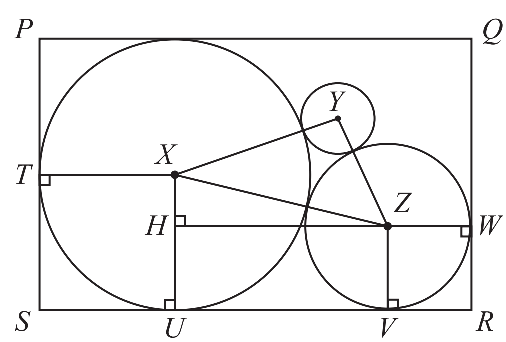
19. Using exponent laws, the expression $\dfrac{2^{x+y}}{2^{x-y}} = 2^{(x+y)-(x-y)} = 2^{2y}$.
Since $x$ and $y$ are positive integers with $xy = 6$, then the possible values of $y$ are the positive divisors of 6, namely 1, 2, 3, or 6. (These correspond to $x = 6, 3, 2, 1$.)
The corresponding values of $2^{2y}$ are $2^2 = 4$, $2^4 = 16$, $2^6 = 64$, and $2^{12} = 4096$.
Therefore, the sum of the possible values of $\dfrac{2^{x+y}}{2^{x-y}}$ is $4 + 16 + 64 + 4096 = 4180$.

ANSWER: (A)

20. Let the radii of the circles with centres $X$, $Y$ and $Z$ be $x$, $y$ and $z$, respectively.
The distance between the centres of two touching circles equals the sum of the radii of these circles.
Therefore, $XY = x + y$ which means that $x + y = 30$.
Also, $XZ = x + z$ which gives $x + z = 40$ and $YZ = y + z$ which gives $y + z = 20$.
Adding these three equations, we obtain $(x + y) + (x + z) + (y + z) = 30 + 40 + 20$ and so $2x + 2y + 2z = 90$ or $x + y + z = 45$.
Since $x + y = 30$ and $x + y + z = 45$, then $30 + z = 45$ and so $z = 15$.
Since $y + z = 20$, then $y = 20 - z = 5$.
Since $x + z = 40$, then $x = 40 - z = 25$.
Knowing the radii of the circles will allow us to calculate the dimensions of the rectangle.

The height of rectangle $PQRS$ equals the "height" of the circle with centre $X$, which is length of the diameter of the circle, or $2x$.
Thus, the height of rectangle $PQRS$ equals 50.

To calculate the width of rectangle $PQRS$, we join $X$ to the points of tangency (that is, the points where the circle touches the rectangle) $T$ and $U$ on $PS$ and $SR$, respectively, $Z$ to the points of tangency $V$ and $W$ on $SR$ and $QR$, respectively, and draw a perpendicular from $Z$ to $H$ on $XU$.
Since radii are perpendicular to tangents at points of tangency, then $XT$, $XU$, $ZV$, and $ZW$ are perpendicular to the sides of the rectangle.




Each of $XTSU$ and $ZWRV$ has three right angles, and so must have four right angles and so are rectangles.
Thus, $SU = TX = 25$ (the radius of the circle with centre $X$) and $VR = ZW = 15$ (the radius of the circle with centre $Z$).
By a similar argument, $HUVZ$ is also a rectangle.
Thus, $UV = HZ$ and $HU = ZV = 15$.
Since $XH = XU - HU$, then $XH = 10$.
By the Pythagorean Theorem, $HZ = \sqrt{XZ^2 - XH^2} = \sqrt{40^2 - 10^2} = \sqrt{1500} = 10\sqrt{15}$ and so $UV = 10\sqrt{15}$.
This means that $SR = SU + UV + VR = 25 + 10\sqrt{15} + 15 = 40 + 10\sqrt{15}$.
Therefore, the area of rectangle $PQRS$ is $50 \times (40 + 10\sqrt{15}) = 2000 + 500\sqrt{15} \approx 3936.5$.
Of the given choices, this answer is closest to (E) 3950.

ANSWER: (E)

21. *Solution 1*

We start with the ones digits.
Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.
Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.
Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.
Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.
Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.
Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.
This gives the following completed multiplication:

$$\begin{array}{r} 1\,0\,2\,5\,6\,4 \\ \times \qquad\quad 4 \\ \hline 4\,1\,0\,2\,5\,6 \end{array}$$

Finally, $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

*Solution 2*

Let $x$ be the five-digit integer with digits $PQRST$.
This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.
Also, $4PQRST = 400\,000 + PQRST = 400\,0000 + x$.
From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives $40x + 16 = 400\,000 + x$ or $39x = 399\,984$.
Thus, $x = \dfrac{399\,984}{39} = 10\,256$.
Since $PQRST = 10\,256$, then $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Answer: (A)

22. Here is one way in which the seven friends can ride on four buses so that the seven restrictions are satisfied:

| Bus 1 | Bus 2 | Bus 3 | Bus 4 |
|---|---|---|---|
| Abu | Bai | Don | Gia |
| Cha | Fan | Eva | |

At least 3 buses are needed because of the groups of 3 friends who must all be on different buses.
We will now show that it is impossible for the 7 friends to travel on only 3 buses.
Suppose that the seven friends could be put on 3 buses.
Since Abu, Bai and Don are on 3 different buses, then we assign them to three buses that we can call Bus 1, Bus 2 and Bus 3, respectively. (See the table below.)
Since Abu, Bai and Eva are on 3 different buses, then Eva must be on Bus 3.
Since Cha and Bai are on 2 different buses and Cha and Eva are on 2 different buses, then Cha cannot be on Bus 2 or Bus 3, so Cha is on Bus 1.
So far, this gives

| Bus 1 | Bus 2 | Bus 3 |
|---|---|---|
| Abu | Bai | Don |
| Cha | | Eva |

The remaining two friends are Fan and Gia.
Since Fan, Cha and Gia are on 3 different buses, then neither Fan nor Gia is on Bus 1.
Since Don, Gia and Fan are on 3 different buses, then neither Fan nor Gia is on Bus 3.
Since Gia and Fan are on separate buses, they cannot both be on Bus 2, which means that the seven friends cannot be on 3 buses only.
Therefore, the minimum number of buses needed is 4.

Answer: (B)

23. Since the wheel turns at a constant speed, then the percentage of time when a shaded part of the wheel touches a shaded part of the path will equal the percentange of the total length of the path where there is "shaded on shaded" contact.
Since the wheel has radius 2 m, then its circumference is $2\pi \times 2$ m which equals $4\pi$ m.
Since the wheel is divided into four quarters, then the portion of the circumference taken by each quarter is $\pi$ m.
We call the left-hand end of the path 0 m.
As the wheel rotates once, the first shaded section of the wheel touches the path between 0 m and $\pi \approx 3.14$ m.
As the wheel continues to rotate, the second shaded section of the wheel touches the path between $2\pi \approx 6.28$ m and $3\pi \approx 9.42$ m.
While the wheel makes 3 complete rotations, a shaded quarter will be in contact with the path over 6 intervals (2 intervals per rotation).
The path is shaded for 1 m starting at each odd multiple of 1 m, and unshaded for 1 m starting at each even multiple of 1 m.
We make a chart of the sections where shaded quarters touch the path and the parts of these intervals that are shaded:

| Beginning of quarter (m) | End of quarter (m) | Shaded parts of path (m) |
|---|---|---|
| 0 | $\pi \approx 3.14$ | 1 to 2; 3 to $\pi$ |
| $2\pi \approx 6.28$ | $3\pi \approx 9.42$ | 7 to 8; 9 to $3\pi$ |
| $4\pi \approx 12.57$ | $5\pi \approx 15.71$ | 13 to 14; 15 to $5\pi$ |
| $6\pi \approx 18.85$ | $7\pi \approx 21.99$ | 19 to 20; 21 to $7\pi$ |
| $8\pi \approx 25.13$ | $9\pi \approx 28.27$ | $8\pi$ to 26; 27 to 28 |
| $10\pi \approx 31.42$ | $11\pi \approx 34.56$ | $10\pi$ to 32; 33 to 34 |

Therefore, the total length of "shaded on shaded", in metres, is

$$1 + (\pi - 3) + 1 + (3\pi - 9) + 1 + (5\pi - 15) + 1 + (7\pi - 21) + (26 - 8\pi) + 1 + (32 - 10\pi) + 1$$

which equals $(16 - 2\pi)$ m.
The total length of the path along which the wheel rolls is $3 \times 4\pi$ m or $12\pi$ m.
This means that the required percentage of time equals $\dfrac{(16 - 2\pi)\text{ m}}{12\pi\text{ m}} \times 100\% \approx 25.8\%$.
Of the given choices, this is closest to 26%, or choice (E).

Answer: (E)

24. We let $A$ be the set $\{2,3,4,5,6,7,8,9\}$.
First, we note that the integer $s$ that Roberta chooses is of the form $s = 11m$ for some integer $m$ from the set $A$, and the integer $t$ that Roberta chooses is of the form $t = 101n$ for some integer $n$ from the set $A$.
This means that the product $rst$ is equal to $r(11m)(101n) = 11 \times 101 \times rmn$ where each of $r, m, n$ comes from the set $A$.
This means that the number of possible values for $rst$ is equal to the number of possible values of $rmn$, and so we count the number of possible values of $rst$ by counting the number of possible values of $rmn$.

We note that $A$ contains only one multiple of 5 and one multiple of 7. Furthermore, these multiples include only one factor of 5 and 7 each, respectively.
We count the number of possible values for $rmn$ by considering the different possibilities for the number of factors of 5 and 7 in $rmn$.
Let $y$ be the number of factors of 5 in $rmn$ and let $z$ be the number of factors of 7 in $rmn$. Note that since each of $r$, $m$ and $n$ includes at most one factor of 5 and at most one factor of 7 and each of $r$, $m$ and $n$ cannot contain both a factor of 5 and a factor of 7, then $y + z$ is at most 3.

<u>Case 1: $y = 3$</u>
If $rmn$ includes 3 factors of 5, then $r = m = n = 5$ and so $rmn = 5^3$.
This means that there is only one possible value for $rmn$.

<u>Case 2: $z = 3$</u>
Here, it must be the case that $rmn = 7^3$ and so there is only one possible value for $rmn$.

<u>Case 3: $y = 2$ and $z = 1$</u>
Here, it must be the case that two of $r, m, n$ are 5 and the other is 7.
In other words, $rmn$ must equal $5^2 \times 7$.
This means that there is only one possible value for $rmn$.

<u>Case 4: $y = 1$ and $z = 2$</u>
Here, $rmn$ must equal $5 \times 7^2$.
This means that there is only one possible value for $rmn$.

<u>Case 5: $y = 2$ and $z = 0$</u>
Here, two of $r, m, n$ must equal 5 and the third cannot be 5 or 7.
This means that the possible values for the third of these are $2, 3, 4, 6, 8, 9$.
This means that there are 6 possible values for $rmn$ in this case.

<u>Case 6: $y = 0$ and $z = 2$</u>
As in Case 5, there are 6 possible values for $rmn$.

<u>Case 7: $y = 1$ and $z = 1$</u>
Here, one of $r, m, n$ equals 5, one equals 7, and the other must be one of $2, 3, 4, 6, 8, 9$.
This means that there are 6 possible values for $rmn$ in this case.

<u>Case 8: $y = 1$ and $z = 0$</u>
Here, one of $r, m, n$ equals 5 and none of these equal 7. Suppose that $r = 5$.
Each of $m$ and $n$ equals one of $2, 3, 4, 6, 8, 9$.

We make a multiplication table to determine the possible values of $mn$:

| $\times$ | 2 | 3 | 4 | 6 | 8 | 9 |
|---|---|---|---|---|---|---|
| 2 | 4 | 6 | 8 | 12 | 16 | 18 |
| 3 | 6 | 9 | 12 | 18 | 24 | 27 |
| 4 | 8 | 12 | 16 | 24 | 32 | 36 |
| 6 | 12 | 18 | 24 | 36 | 48 | 54 |
| 8 | 16 | 24 | 32 | 48 | 64 | 72 |
| 9 | 18 | 27 | 36 | 54 | 72 | 81 |

The different values in this table are

$$4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81$$

of which there are 16.
Therefore, there are 16 possible values of $rmn$ in this case.

Case 9: $y = 0$ and $z = 1$
As in Case 8, there are 16 possible values of $rmn$.

Case 10: $y = 0$ and $z = 0$
Here, none of $r, m, n$ is 5 or 7.
This means that each of $r, m, n$ equals one of $2, 3, 4, 6, 8, 9$.
This means that the only possible prime factors of $rmn$ are 2 and 3.
Each of $2, 3, 4, 6, 8, 9$ includes at most 2 factors of 3 and only 9 includes 2 factors of 3.
This means that $rmn$ contains at most 6 factors of 3.
Let $w$ be the number of factors of 3 in $rmn$.

- If $w = 6$, then $r = m = n = 9$ and so $rmn = 9^3$. There is one value of $rmn$ here.
- If $w = 5$, then two of $r, m, n$ equal 9 and the third is 3 or 6. This means that $rmn = 9^2 \times 3$ or $rmn = 9^2 \times 6$. There are two values of $rmn$ here.
- If $w = 4$, then either two of $r, m, n$ equal 9 and the third does not include a factor of 3, or one of $r, m, n$ equals 9 and the second and third are each 3 or 6.
  Thus, $rmn$ can equal $9^2 \times 2$, $9^2 \times 4$, $9^2 \times 8$, $9 \times 3 \times 3$, $9 \times 3 \times 6$, or $9 \times 6 \times 6$.
  There is duplication in this list and so the values of $rmn$ are $9^2$, $9^2 \times 2$, $9^2 \times 4$, $9^2 \times 8$.
  There are four values of $rmn$ here.
- If $w = 3$, we can have one of $r, m, n$ equal to 9, one equal to 3 or 6, and the last equal to 2, 4 or 8, or we can have each of $r, m, n$ equal to 3 or 6.
  In the first situation, $rmn$ can be $9 \times 3 \times 2$, $9 \times 3 \times 4$, $9 \times 3 \times 8$, $9 \times 6 \times 2$, $9 \times 6 \times 4$, $9 \times 6 \times 8$.
  These can be written as $27 \times 2^1$, $27 \times 2^2$, $27 \times 2^3$, $27 \times 2^4$.
  In the second situation, $rmn$ can be $3 \times 3 \times 3$, $3 \times 3 \times 6$, $3 \times 6 \times 6$, $6 \times 6 \times 6$.
  These equal $27$, $27 \times 2^1$, $27 \times 2^2$, $27 \times 2^3$.
  Combining lists, we get $27$, $27 \times 2^1$, $27 \times 2^2$, $27 \times 2^3$, $27 \times 2^4$.
  Therefore, in this case there are 5 possible values for $rmn$.
- If $w = 2$, then either one of $r, m, n$ equals 9 or two of $r, m, n$ equal 3 or 6.
  In the first situation, the other two of $r, m, n$ equal 2, 4 or 8.
  Note that $2 = 2^1$ and $4 = 2^2$ and $8 = 2^3$.
  Thus $rmn$ contains at least 2 factors of 2 (for example, $rmn = 9 \times 2 \times 2$) and contains at most 6 factors of 2 ($rmn = 9 \times 8 \times 8$).

If two of $r, m, n$ equal 3 or 6, then the third equals 2, 4 or 8.
Thus, $rmn$ contains at least 1 factor of 2 ($rmn = 3 \times 3 \times 2$) and at most 5 factors of 2 ($rmn = 6 \times 6 \times 8$).
Combining these possibilities, $rmn$ can equal $9 \times 2, 9 \times 2^2, \ldots, 9 \times 2^6$, and so there are 6 possible values of $rmn$ in this case.

- If $w = 1$, then one of $r, m, n$ equals 3 or 6, and the other two equal 2, 4 or 8.
  Thus, $rmn$ can contain at most 7 factors of 2 ($rmn = 6 \times 8 \times 8$) and must contain at least 2 factors of 2 ($rmn = 3 \times 2 \times 2$).
  Each of the number of factors of 2 between 2 and 7, inclusive, is possible, so there are 6 possible values of $rmn$.
- If $w = 0$, then none of $r, m, n$ can equal 3, 6 or 9.
  Thus, each of $r, m, n$ equals $2^1$, $2^2$ or $2^3$.
  Thus, $rmn$ must be a power of 2 and includes at least 3 and no more than 9 factors of 2. Each of these is possible, so there are 7 possible values of $rmn$.

In total, the number of possible values of $rmn$ (and hence of $rst$) is

$$2 \times 1 + 2 \times 1 + 2 \times 6 + 6 + 2 \times 16 + (1 + 2 + 4 + 5 + 6 + 6 + 7)$$

which equals 85.

Answer: (A)

25. Suppose that $PQ = a$, $PS = b$ and $PU = c$.
Since $PQRSTUVW$ is a rectangular prism, then $QR = PS = b$ and $ST = QV = PU = c$.
By the Pythagorean Theorem, $PR^2 = PQ^2 + QR^2$ and so $1867^2 = a^2 + b^2$.
By the Pythagorean Theorem, $PV^2 = PQ^2 + QV^2$ and so $2019^2 = a^2 + c^2$.
By the Pythagorean Theorem, $PT^2 = PS^2 + ST^2$ and so $x^2 = b^2 + c^2$.
Adding these three equations, we obtain

$$\begin{aligned} 1867^2 + 2019^2 + x^2 &= (a^2 + b^2) + (a^2 + c^2) + (b^2 + c^2) \\ 1867^2 + 2019^2 + x^2 &= 2a^2 + 2b^2 + 2c^2 \\ a^2 + b^2 + c^2 &= \frac{1867^2 + 2019^2 + x^2}{2} \end{aligned}$$

Since $a^2 + b^2 = 1867^2$, then

$$c^2 = (a^2 + b^2 + c^2) - (a^2 + b^2) = \frac{1867^2 + 2019^2 + x^2}{2} - 1867^2 = \frac{-1867^2 + 2019^2 + x^2}{2}$$

Since $a^2 + c^2 = 2019^2$, then

$$b^2 = (a^2 + b^2 + c^2) - (a^2 + c^2) = \frac{1867^2 + 2019^2 + x^2}{2} - 2019^2 = \frac{1867^2 - 2019^2 + x^2}{2}$$

Since $b^2 + c^2 = x^2$, then

$$a^2 = (a^2 + b^2 + c^2) - (b^2 + c^2) = \frac{1867^2 + 2019^2 + x^2}{2} - x^2 = \frac{1867^2 + 2019^2 - x^2}{2}$$

Since $a$, $b$ and $c$ are edge lengths of the prism, then $a, b, c > 0$.
Since $a^2 > 0$, then $\dfrac{1867^2 + 2019^2 - x^2}{2} > 0$ and so $1867^2 + 2019^2 - x^2 > 0$ or $x^2 < 2019^2 + 1867^2$.
Since $b^2 > 0$, then $\dfrac{1867^2 - 2019^2 + x^2}{2} > 0$ and so $1867^2 - 2019^2 + x^2 > 0$ or $x^2 > 2019^2 - 1867^2$.
Since $c^2 > 0$, then $\dfrac{-1867^2 + 2019^2 + x^2}{2} > 0$ and so $-1867^2 + 2019^2 + x^2 > 0$ which means that $x^2 > 1867^2 - 2019^2$.
Since the right side of this last inequality is negative and the left side is non-negative, then this inequality is always true.
Therefore, it must be true that $2019^2 - 1867^2 < x^2 < 2019^2 + 1867^2$.
Since all three parts of this inequality are positive, then $\sqrt{2019^2 - 1867^2} < x < \sqrt{2019^2 + 1867^2}$.
Since $\sqrt{2019^2 - 1867^2} \approx 768.55$ and $\sqrt{2019^2 + 1867^2} \approx 2749.92$ and $x$ is an integer, then $769 \leq x \leq 2749$.
The number of integers $x$ in this range is $2749 - 769 + 1 = 1981$.
Every such value of $x$ gives positive values for $a^2$, $b^2$ and $c^2$ and so positive values for $a$, $b$ and $c$, and so a rectangular prism $PQRSTUVW$ with the correct lengths of face diagonals.
Therefore, there are 1981 such integers $x$.

Answer: (E)