The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2019 Canadian Senior Mathematics Contest

Wednesday, November 20, 2019
(in North America and South America)

Thursday, November 21, 2019
(outside of North America and South America)

Solutions

Part A

1. Since Zipporah is 7 years old and the sum of Zipporah's age and Dina's age is 51, then Dina is $51 - 7 = 44$ years old.
   Since Dina is 44 years old and the sum of Julio's age and Dina's age is 54, then Julio is $54 - 44 = 10$ years old.

   ANSWER: 10

2. Since the circular track has radius 60 m, its circumference is $2\pi \cdot 60$ m which equals 120π m.
   Since Ali runs around this track at a constant speed of 6 m/s, then it takes Ali $\dfrac{120\pi \text{ m}}{6 \text{ m/s}} = 20\pi$ s to complete one lap.
   Since Ali and Darius each complete one lap in the same period of time, then Darius also takes 20π s to complete one lap.
   Since Darius runs at a constant speed of 5 m/s, then the length of his track is $20\pi \text{ s} \cdot 5 \text{ m/s}$ or 100π m.
   Since Darius's track is in the shape of an equilateral triangle with side length x m, then its perimeter is $3x$ m and so $3x \text{ m} = 100\pi$ m and so $x = \frac{100\pi}{3}$.

   ANSWER: $x = \frac{100\pi}{3}$

3. Since $2^a \cdot 2^b = 2^{a+b}$, then

$$\begin{aligned} 32^n &= 2^{200} \cdot 2^{203} + 2^{163} \cdot 2^{241} + 2^{126} \cdot 2^{277} \\ &= 2^{200+203} + 2^{163+241} + 2^{126+277} \\ &= 2^{403} + 2^{404} + 2^{403} \\ &= 2^{403} + 2^{403} + 2^{404} \end{aligned}$$

   Since $2^c + 2^c = 2(2^c) = 2^1 \cdot 2^c = 2^{c+1}$, then

$$\begin{aligned} 32^n &= 2^{403+1} + 2^{404} \\ &= 2^{404} + 2^{404} \\ &= 2^{404+1} \\ &= 2^{405} \end{aligned}$$

   Since $(2^d)^e = 2^{de}$, then $32^n = (2^5)^n = 2^{5n}$.
   Since $32^n = 2^{405}$, then $2^{5n} = 2^{405}$ which means that $5n = 405$ and so $n = 81$.

   ANSWER: $n = 81$

4. For there to exist a pair of integers (x, y) with $x^2 \leq y \leq x + 6$, it must be the case that $x^2 \leq x + 6$ and so $x^2 - x - 6 \leq 0$.
Now $x^2 - x - 6 = (x - 3)(x + 2)$, so $x^2 - x - 6 \leq 0$ exactly when $-2 \leq x \leq 3$. (If we consider the function $f(x) = (x - 3)(x + 2)$, whose graph is a parabola opening upwards, its values are less than or equal to 0 between its roots.)
Therefore, any pair of integers (x, y) with $x^2 \leq y \leq x + 6$ must have x equal to one of $-2, -1, 0, 1, 2, 3$.
When $x = -2$, the original inequality becomes $4 \leq y \leq 4$ and so y must equal 4. There is 1 pair in this case, namely $(-2, 4)$.
When $x = -1$, we obtain $1 \leq y \leq 5$ and so y must equal one of $1, 2, 3, 4, 5$. There are 5 pairs in this case.
When $x = 0$, we obtain $0 \leq y \leq 6$ and so y must equal one of $0, 1, 2, 3, 4, 5, 6$. There are 7 pairs in this case.
When $x = 1$, we obtain $1 \leq y \leq 7$. There are 7 pairs in this case.
When $x = 2$, we obtain $4 \leq y \leq 8$. There are 5 pairs in this case.
When $x = 3$, we obtain $9 \leq y \leq 9$ and so y must equal 9. There is 1 pair in this case.
In total, there are $1 + 5 + 7 + 7 + 5 + 1 = 26$ pairs of integers that satisfy the inequality.

Answer: 26

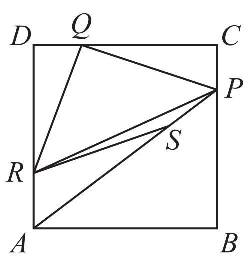
5. Since 605 is the middle side length of the right-angled triangle, we suppose that the side lengths of the triangle are $a, 605, c$ for integers $a < 605 < c$. (Why do we not need to consider the cases $a = 605$ or $605 = c$?)
By the Pythagorean Theorem, knowing that c (the longest side length) must be the length of the hypotenuse, we obtain $a^2 + 605^2 = c^2$ and so $c^2 - a^2 = 605^2$.
We want to determine the maximum possible length of the shortest side of the triangle.
In other words, we want to try to determine the maximum possible length of a which is less than 605.
We note that $c^2 - a^2 = 605^2$ exactly when $(c + a)(c - a) = 605^2$.
We note also that $605 = 5 \cdot 121 = 5 \cdot 11^2$ and so $605^2 = 5^2 \cdot 11^4$.
Therefore, we have $(c + a)(c - a) = 5^2 \cdot 11^4$. This means that $c + a$ and $c - a$ are a divisor pair of $5^2 \cdot 11^4$.
Since a and c are positive integers, then $c + a > c - a$. Note that $c > a$ and so $c + a > c - a > 0$.
We make a table of the possible values for $c + a$ and $c - a$, and use these to determine the possible values of c and a

$c + a$	$c - a$	$2c = (c + a) + (c - a)$	c	$a = (c + a) - c$
$5^2 \cdot 11^4 = 366025$	1	366026	183013	103012
$5 \cdot 11^4 = 73205$	5	73210	36605	36600
$5^2 \cdot 11^3 = 33275$	11	33286	16643	16632
$11^4 = 14641$	$5^2 = 25$	14666	7333	7308
$5 \cdot 11^3 = 6655$	$5 \cdot 11 = 55$	6710	3355	3300
$5^2 \cdot 11^2 = 3025$	$11^2 = 121$	3146	1573	1452
$11^3 = 1331$	$5^2 \cdot 11 = 275$	1606	803	528
$5 \cdot 11^2 = 605$	$5 \cdot 11^2 = 605$	1210	605	0

These are all of the possible factorizations of 605^2, and so give all of the possible pairs (a, c) that satisfy the equation.
Therefore, the maximum possible value of a that is less than 605 is 528.

Answer: 528

6. Since square $ABCD$ has side length 4, then its area is 4^2, which equals 16.
The area of quadrilateral $PQRS$, which we expect to be a function of k, equals the area of square $ABCD$ minus the combined areas of $\triangle ABP$, $\triangle PCQ$, $\triangle QDR$, and $\triangle ARS$.
Since $\dfrac{BP}{PC} = \dfrac{k}{4-k}$, then there is a real number x with $BP = kx$ and $PC = (4-k)x$.
Since $BP + PC = BC = 4$, then $kx + (4-k)x = 4$ and so $4x = 4$ or $x = 1$.
Thus, $BP = k$ and $PC = 4 - k$.
Similarly, $CQ = DR = k$ and $QD = RA = 4 - k$.
$\triangle ABP$ is right-angled at B and so its area is $\frac{1}{2}(AB)(BP) = \frac{1}{2}(4k) = 2k$.
$\triangle PCQ$ is right-angled at C and so its area is $\frac{1}{2}(PC)(CQ) = \frac{1}{2}(4-k)k$.
$\triangle QDR$ is right-angled at D and so its area is $\frac{1}{2}(QD)(DR) = \frac{1}{2}(4-k)k$.
To find the area of $\triangle ARS$, we first join R to P.



Now $\triangle ARP$ can be seen as having base $RA = 4 - k$ and perpendicular height equal to the distance between the parallel lines CB and DA, which equals 4.
Thus, the area of $\triangle ARP$ is $\frac{1}{2}(4-k)(4)$.
Now we consider $\triangle ARP$ as having base AP divided by point S in the ratio $k : (4-k)$.
This means that the ratio of $AS : AP$ equals $k : ((4-k)+k)$ which equals $k : 4$.
This means that the area of $\triangle ARS$ is equal to $\dfrac{k}{4}$ times the area of $\triangle ARP$. (The two triangles have the same height – the distance from R to AP – and so the ratio of their areas equals the ratio of their bases.)
Thus, the area of $\triangle ARS$ equals $\dfrac{\frac{1}{2}(4-k)(4)\cdot k}{4} = \frac{1}{2}k(4-k)$.
Thus, the area of quadrilateral $PQRS$ is

$$\begin{aligned} 16 - 2k - 3\cdot\tfrac{1}{2}k(4-k) &= 16 - 2k - \tfrac{3}{2}\cdot 4k + \tfrac{3}{2}k^2 \\ &= \tfrac{3}{2}k^2 - 2k - 6k + 16 \\ &= \tfrac{3}{2}k^2 - 8k + 16 \end{aligned}$$

The minimum value of the quadratic function $f(t) = at^2 + bt + c$ with $a > 0$ occurs when $t = -\dfrac{b}{2a}$ and so the minimum value of $\frac{3}{2}k^2 - 8k + 16$ occurs when $k = -\frac{-8}{2(3/2)} = \frac{8}{3}$.
Therefore, the area of quadrilateral $PQRS$ is minimized when $k = \frac{8}{3}$.

Answer: $k = \frac{8}{3}$

Part B

1. (a) Since each of Rachel's jumps is 168 cm long, then when Rachel completes 5 jumps, she jumps 5×168 cm $= 840$ cm.
Since each of Joel's jumps is 120 cm long, then when Joel completes n jumps, he jumps $120n$ cm.
Since Rachel and Joel jump the same total distance, then $120n = 840$ and so $n = 7$.

(b) Since each of Joel's jumps is 120 cm long, then when Joel completes r jumps, he jumps $120r$ cm.
Since each of Mark's jumps is 72 cm long, then when Mark completes t jumps, he jumps $72t$ cm.
Since Joel and Mark jump the same total distance, then $120r = 72t$ and so dividing by 24, $5r = 3t$.
Since $5r$ is a multiple of 5, then $3t$ must also be a multiple of 5, which means that t is a multiple of 5.
Since $11 \leq t \leq 19$ and t is a multiple of 5, then $t = 15$.
Since $t = 15$, then $5r = 3 \cdot 15 = 45$ and so $r = 9$.
Therefore, $r = 9$ and $t = 15$.

(c) When Rachel completes a jumps, she jumps $168a$ cm.
When Joel completes b jumps, he jumps $120b$ cm.
When Mark completes c jumps, he jumps $72c$ cm.
Since Rachel, Joel and Mark all jump the same total distance, then $168a = 120b = 72c$.
Dividing by 24, we obtain $7a = 5b = 3c$.
Since $7a$ is divisible by 7, then $3c$ is divisible by 7, which means that c is divisible by 7.
Since $5b$ is divisible by 5, then $3c$ is divisible by 5, which means that c is divisible by 5.
Since c is divisible by 5 and by 7 and because 5 and 7 have no common divisor larger than 1, then c must be divisible by $5 \cdot 7$ which equals 35.
Since c is divisible by 35 and c is a positive integer, then $c \geq 35$.
We note that if $c = 35$, then $3c = 105$ and since $7a = 5b = 105$, we obtain $a = 15$ and $b = 21$. In other words, $c = 35$ is possible.
Therefore, the minimum possible value of c is $c = 35$.

2. (a) For the sequence $\frac{1}{w}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ to be an arithmetic sequence, it must be the case that

$$\frac{1}{2} - \frac{1}{w} = \frac{1}{3} - \frac{1}{2} = \frac{1}{6} - \frac{1}{3}$$

Since $\frac{1}{3} - \frac{1}{2} = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6}$, then $\frac{1}{2} - \frac{1}{w} = -\frac{1}{6}$ and so $\frac{1}{w} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, which gives $w = \frac{3}{2}$.

(b) The sequence $\frac{1}{y+1}, x, \frac{1}{z+1}$ is arithmetic exactly when $x - \frac{1}{y+1} = \frac{1}{z+1} - x$ or $2x = \frac{1}{y+1} + \frac{1}{z+1}$.

Since $y, 1, z$ is a geometric sequence, then $\frac{1}{y} = \frac{z}{1}$ and so $z = \frac{1}{y}$. Since y and z are positive, then $y \neq -1$ and $z \neq -1$.

In this case, $\frac{1}{y+1} + \frac{1}{z+1} = \frac{1}{y+1} + \frac{1}{\frac{1}{y}+1} = \frac{1}{y+1} + \frac{y}{1+y} = \frac{y+1}{y+1} = 1$.

Since $\frac{1}{y+1} + \frac{1}{z+1} = 1$, then the sequence $\frac{1}{y+1}, x, \frac{1}{z+1}$ is arithmetic exactly when $2x = 1$ or $x = \frac{1}{2}$.

(c) Since a, b, c, d is a geometric sequence, then $b = ar$, $c = ar^2$ and $d = ar^3$ for some real number r. Since $a \neq b$, then $a \neq 0$. (If $a = 0$, then $b = 0$.)

Since $a \neq b$, then $r \neq 1$. Note that $\frac{b}{a} = \frac{ar}{a} = r$ and so we want to determine all possible values of r.

Since a and b are both positive, then $r > 0$.

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{d}$ is an arithmetic sequence, then

$$\begin{aligned}
\frac{1}{b} - \frac{1}{a} &= \frac{1}{d} - \frac{1}{b} \\
\frac{1}{ar} - \frac{1}{a} &= \frac{1}{ar^3} - \frac{1}{ar} \\
\frac{1}{r} - 1 &= \frac{1}{r^3} - \frac{1}{r} \qquad \text{(since } a \neq 0\text{)} \\
r^2 - r^3 &= 1 - r^2 \\
0 &= r^3 - 2r^2 + 1 \\
0 &= (r-1)(r^2 - r - 1)
\end{aligned}$$

Since $r \neq 1$, then $r^2 - r - 1 = 0$.

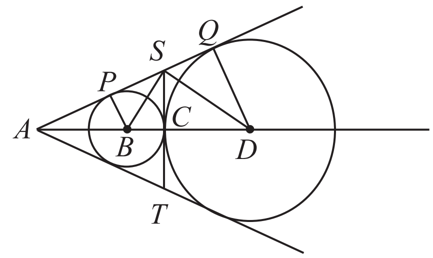
By the quadratic formula, $r = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

Since a and b are both positive, then $r > 0$ and so $r = \frac{1+\sqrt{5}}{2}$.

This is the only possible value of r.

We can check that r satisfies the conditions by verifying that when $a = 1$ (for example) and $r = \frac{1+\sqrt{5}}{2}$, giving $b = \frac{1+\sqrt{5}}{2}$, $c = \left(\frac{1+\sqrt{5}}{2}\right)^2$, and $d = \left(\frac{1+\sqrt{5}}{2}\right)^3$, then we do indeed obtain $\frac{1}{b} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}$.

3. (a) Since $AS = ST = AT$, then $\triangle AST$ is equilateral.
This means that $\angle TAS = \angle AST = \angle ATS = 60^\circ$.
Join B to P, B to S, D to Q and D to S.



Since AS is tangent to the circle with centre B at P, then BP is perpendicular to PS.
Since BP and BC are radii of the circle with centre B, then $BP = BC = 1$.
Consider $\triangle SBP$ and $\triangle SBC$.
Each is right-angled (at P and C), they have a common hypotenuse BS, and equal side lengths ($BP = BC$).
This means that $\triangle SBP$ and $\triangle SBC$ are congruent.
Thus, $\angle PSB = \angle CSB = \frac{1}{2}\angle AST = 30^\circ$.
This means that $\triangle SBC$ is a 30°-60°-90° triangle, and so $SC = \sqrt{3}BC = \sqrt{3}$.
Since $\angle CSQ = 180^\circ - \angle CSP = 180^\circ - 60^\circ = 120^\circ$, then using a similar argument we can see that $\triangle DSC$ is also a 30°-60°-90° triangle.
This means that $CD = \sqrt{3}SC = \sqrt{3} \cdot \sqrt{3} = 3$.
Since CD is a radius of the circle with centre D, then $r = CD = 3$.

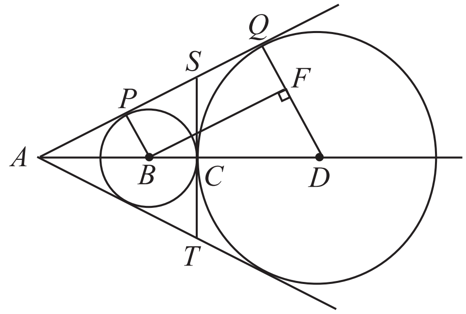
(b) *Solution 1*
From the given information, $DQ = QP = r$.
Again, join B to P, B to S, D to Q, and D to S.
As in (a), $\triangle SBP$ and $\triangle SBC$ are congruent which means that $SP = SC$.
Using a similar argument, $\triangle SDC$ is congruent to $\triangle SDQ$.
This means that $SC = SQ$.
Since $SP = SC$ and $SC = SQ$, then $SP = SQ$.
Since $QP = r$, then $SP = SQ = \frac{1}{2}r$.
Suppose that $\angle PSC = 2\theta$.
Since $\triangle SBP$ and $\triangle SBC$ are congruent, then $\angle PSB = \angle CSB = \frac{1}{2}\angle PSC = \theta$.
Since $\angle QSC = 180^\circ - \angle PSC = 180^\circ - 2\theta$, then $\angle QSD = \angle CSD = \frac{1}{2}\angle QSC = 90^\circ - \theta$.
Since $\triangle SDQ$ is right-angled at Q, then $\angle SDQ = 90^\circ - \angle QSD = \theta$.
This means that $\triangle SBP$ is similar to $\triangle DSQ$.
Therefore, $\dfrac{SP}{BP} = \dfrac{DQ}{SQ}$ and so $\dfrac{\frac{1}{2}r}{1} = \dfrac{r}{\frac{1}{2}r} = 2$, which gives $\frac{1}{2}r = 2$ and so $r = 4$.

Solution 2

From the given information, $DQ = QP = r$.

Join B to P and D to Q. As in (a), BP and DQ are perpendicular to PQ.

Join B to F on QD so that BF is perpendicular to QD.



This means that $\triangle BFD$ is right-angled at F.

Also, since $BPQF$ has three right angles, then it must have four right angles and so is a rectangle.

Thus, $BF = PQ = r$ and $QF = PB = 1$.

Since $QD = r$, then $FD = r - 1$.

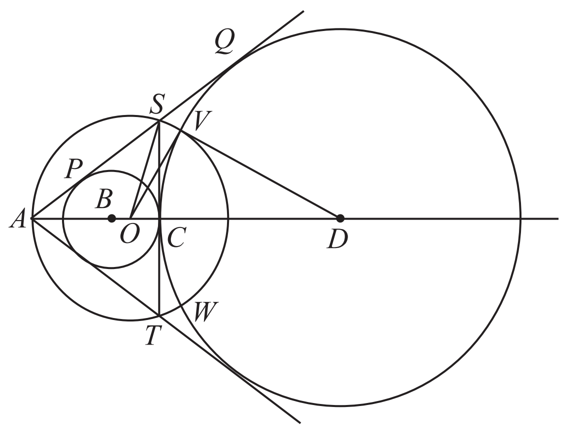
Also, $BD = BC + CD = 1 + r$.

Using the Pythagorean Theorem in $\triangle BFD$, we obtain the following equivalent equations:

$$BF^2 + FD^2 = BD^2$$
$$r^2 + (r-1)^2 = (r+1)^2$$
$$r^2 + r^2 - 2r + 1 = r^2 + 2r + 1$$
$$r^2 = 4r$$

Since $r \neq 0$, then it must be the case that $r = 4$.

(c) As in Solution 1 to (b), $\triangle SBP$ is similar to $\triangle DSQ$ and $SP = SQ$. Therefore, $\dfrac{SP}{BP} = \dfrac{DQ}{SQ}$ or $\dfrac{SP}{1} = \dfrac{r}{SP}$ which gives $SP^2 = r$ and so $SP = \sqrt{r}$.
Thus, $SP = SQ = SC = \sqrt{r}$.
Next, $\triangle APB$ is similar to $\triangle AQD$ (common angle at A, right angle).
Therefore, $\dfrac{AB}{BP} = \dfrac{AD}{DQ}$ and so $\dfrac{AB}{1} = \dfrac{AB + BD}{r}$ and so $AB = \dfrac{AB + 1 + r}{r}$.
Re-arranging gives $rAB = AB + 1 + r$ and so $(r-1)AB = r + 1$ and so $AB = \dfrac{r+1}{r-1}$.
This means that $AC = AB + BC = AB + 1 = \dfrac{r+1}{r-1} + 1 = \dfrac{(r+1)+(r-1)}{r-1} = \dfrac{2r}{r-1}$.
Next, draw the circle with centre O that passes through A, S and T and through point V on the circle with centre D so that OV is perpendicular to DV.



Let the radius of this circle be R. Note that $OS = AO = R$.
Consider $\triangle OSC$.
This triangle is right-angled at C.
Using the Pythagorean Theorem, we obtain the following equivalent equations:

$$\begin{aligned}
OS^2 &= OC^2 + SC^2 \\
R^2 &= (AC - AO)^2 + SC^2 \\
R^2 &= (AC - R)^2 + SC^2 \\
R^2 &= AC^2 - 2R \cdot AC + R^2 + SC^2 \\
2R \cdot AC &= AC^2 + SC^2 \\
R &= \frac{AC}{2} + \frac{SC^2}{2AC} \\
R &= \frac{2r}{2(r-1)} + \frac{(\sqrt{r})^2}{4r/(r-1)} \\
R &= \frac{r}{r-1} + \frac{r-1}{4}
\end{aligned}$$

Since OV is perpendicular to DV, then $\triangle OVD$ is right-angled at V.

Using the Pythagorean Theorem, noting that $OV = R$ and $DV = r$, we obtain the following equivalent equations:

$$\begin{aligned}
OV^2 + DV^2 &= OD^2 \\
R^2 + r^2 &= (OC + CD)^2 \\
R^2 + r^2 &= (AC - AO + CD)^2 \\
R^2 + r^2 &= \left(\frac{2r}{r-1} - R + r\right)^2 \\
R^2 + r^2 &= \left(\frac{2r + r(r-1)}{r-1} - R\right)^2 \\
R^2 + r^2 &= \left(\frac{r^2 + r}{r-1} - R\right)^2 \\
R^2 + r^2 &= \left(\frac{r^2 + r}{r-1}\right)^2 - 2R\left(\frac{r^2 + r}{r-1}\right) + R^2 \\
2R\left(\frac{r^2 + r}{r-1}\right) &= \left(\frac{r^2 + r}{r-1}\right)^2 - r^2 \\
2R\left(\frac{r(r+1)}{r-1}\right) &= \frac{r^2(r+1)^2}{(r-1)^2} - r^2 \\
2R &= \frac{r-1}{r(r+1)} \cdot \frac{r^2(r+1)^2}{(r-1)^2} - \frac{r-1}{r(r+1)} \cdot r^2 \\
2R &= \frac{r(r+1)}{r-1} - \frac{r(r-1)}{r+1}
\end{aligned}$$

Since $R = \dfrac{r}{r-1} + \dfrac{r-1}{4}$, we obtain:

$$\frac{2r}{r-1} + \frac{r-1}{2} = \frac{r(r+1)}{r-1} - \frac{r(r-1)}{r+1}$$

Multiplying both sides by $2(r+1)(r-1)$, expanding, simplifying, and factoring, we obtain the following equivalent equations:

$$\begin{aligned}
4r(r+1) + (r-1)^2(r+1) &= 2r(r+1)^2 - 2r(r-1)^2 \\
(4r^2 + 4r) + (r-1)(r^2-1) &= 2r((r+1)^2 - (r-1)^2) \\
(4r^2 + 4r) + (r^3 - r^2 - r + 1) &= 2r((r^2 + 2r + 1) - (r^2 - 2r + 1)) \\
(4r^2 + 4r) + (r^3 - r^2 - r + 1) &= 2r(4r) \\
r^3 - 5r^2 + 3r + 1 &= 0 \\
(r-1)(r^2 - 4r - 1) &= 0
\end{aligned}$$

Now $r \neq 1$. (If $r = 1$, the circles would be the same size and the two common tangents would be parallel.)
Therefore, $r \neq 1$ which means that $r^2 - 4r - 1 = 0$.
By the quadratic formula,

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

Since $r > 1$, then $r = 2 + \sqrt{5}$.