The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *Canadian Senior Mathematics Contest*

***Wednesday, November 20, 2019***
***(in North America and South America)***

***Thursday, November 21, 2019***
***(outside of North America and South America)***

UNIVERSITY OF WATERLOO

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**Time:** 2 hours 

**Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.**

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

**PART A**

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

**PART B**

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

**At the completion of the contest, insert your student information form inside your answer booklet.**

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*Do not discuss the problems or solutions from this contest online for the next 48 hours.*

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*The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.*

## *Canadian Senior Mathematics Contest*

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the $x$-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

# PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. The sum of Zipporah's age and Dina's age is 51. The sum of Julio's age and Dina's age is 54. Zipporah is 7 years old. How old is Julio?

2. A circular track has a radius of 60 m. Ali runs around the circular track at a constant speed of 6 m/s. A track in the shape of an equilateral triangle has a side length of $x$ m. Darius runs around this triangular track at a constant speed of 5 m/s. Ali and Darius each complete one lap in exactly the same amount of time. What is the value of $x$?

3. If $2^{200} \cdot 2^{203} + 2^{163} \cdot 2^{241} + 2^{126} \cdot 2^{277} = 32^n$, what is the value of $n$?

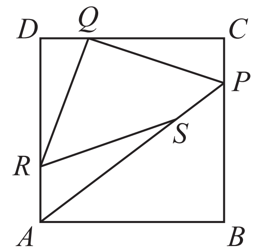
4. How many ordered pairs of integers $(x, y)$ satisfy $x^2 \leq y \leq x + 6$ ?

5. A right-angled triangle with integer side lengths has one side with length 605. This side is neither the shortest side nor the longest side of the triangle. What is the maximum possible length of the shortest side of this triangle?

6. Suppose that $ABCD$ is a square with side length 4 and that $0 < k < 4$. Let points $P$, $Q$, $R$, and $S$ be on $BC$, $CD$, $DA$, and $AP$, respectively, so that

$$\frac{BP}{PC} = \frac{CQ}{QD} = \frac{DR}{RA} = \frac{AS}{SP} = \frac{k}{4-k}$$

What is the value of $k$ which minimizes the area of quadrilateral $PQRS$?



# PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. Rachel does jumps, each of which is 168 cm long. Joel does jumps, each of which is 120 cm long. Mark does jumps, each of which is 72 cm long.

   (a) On Monday, Rachel completes 5 jumps and Joel completes $n$ jumps. Rachel and Joel jump the same total distance. Determine the value of $n$.

   (b) On Tuesday, Joel completes $r$ jumps and Mark completes $t$ jumps. Joel and Mark jump the same total distance, and $r$ and $t$ are integers. If $11 \leq t \leq 19$, determine the values of $r$ and $t$.

   (c) On Wednesday, Rachel completes $a$ jumps, Joel completes $b$ jumps, and Mark completes $c$ jumps. Each of $a$, $b$ and $c$ is a positive integer, and Rachel, Joel and Mark each jump the same total distance. Determine the minimum possible value of $c$ and explain why this value is indeed the minimum.

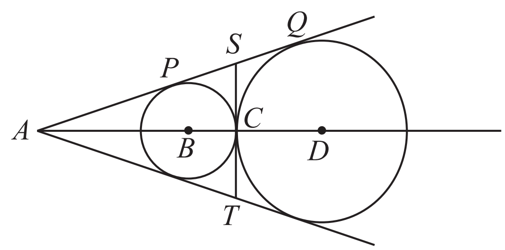
2. An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ is an arithmetic sequence with four terms.
   A *geometric sequence* is a sequence in which each term after the first is obtained by multiplying the previous term by a constant. For example, $3, 6, 12, 24$ is a geometric sequence with four terms.

   (a) Determine a real number $w$ for which $\frac{1}{w}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ is an arithmetic sequence.

   (b) Suppose $y, 1, z$ is a geometric sequence with $y$ and $z$ both positive. Determine all real numbers $x$ for which $\frac{1}{y+1}, x, \frac{1}{z+1}$ is an arithmetic sequence for all such $y$ and $z$.

   (c) Suppose that $a, b, c, d$ is a geometric sequence and $\frac{1}{a}, \frac{1}{b}, \frac{1}{d}$ is an arithmetic sequence with each of $a$, $b$, $c$, and $d$ positive and $a \neq b$. Determine all possible values of $\frac{b}{a}$.

3. In the diagram, the circles with centres $B$ and $D$ have radii 1 and $r$, respectively, and are tangent at $C$. The line through $A$ and $D$ passes through $B$. The line through $A$ and $S$ is tangent to the circles with centres $B$ and $D$ at $P$ and $Q$, respectively. The line through $A$ and $T$ is also tangent to both circles. Line segment $ST$ is perpendicular to $AD$ at $C$ and is tangent to both circles at $C$.




(a) There is a value of $r$ for which $AS = ST = AT$. Determine this value of $r$.

(b) There is a value of $r$ for which $DQ = QP$. Determine this value of $r$.

(c) A third circle, with centre $O$, passes through $A$, $S$ and $T$, and intersects the circle with centre $D$ at points $V$ and $W$. There is a value of $r$ for which $OV$ is perpendicular to $DV$. Determine this value of $r$.

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