The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Pascal Contest

(Grade 9)

Tuesday, February 27, 2018
(in North America and South America)

Wednesday, February 28, 2018
(outside of North America and South America)

Solutions

1. When we arrange the five choices from smallest to largest, we obtain $1.2, 1.4, 1.5, 2.0, 2.1$. Thus, 1.2 is the smallest.

ANSWER: (B)

2. Evaluating, $\dfrac{2018 - 18 + 20}{2} = \dfrac{2000 + 20}{2} = \dfrac{2020}{2} = 1010$.

ANSWER: (A)

3. July 14 is 11 days after July 3 of the same year.
Since there are 7 days in a week, then July 10 and July 3 occur on the same day of the week, namely Wednesday.
July 14 is 4 days after July 10, and so is a Sunday.

ANSWER: (C)

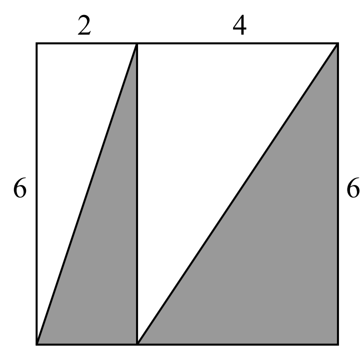
4. Since the car is charged 3 times per week for 52 weeks, it is charged $3 \times 52 = 156$ times.
Since the cost per charge is \$0.78, then the total cost is $156 \times \$0.78 = \$121.68$.

ANSWER: (E)

5. Since
$$3 \times 3 \times 5 \times 5 \times 7 \times 9 = 3 \times 3 \times 7 \times n \times n$$
then
$$n \times n = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 9}{3 \times 3 \times 7} = 5 \times 5 \times 9 = 5 \times 5 \times 3 \times 3$$
Since $n \times n = 5 \times 5 \times 3 \times 3$, then a possible value for n is $n = 5 \times 3 = 15$.

ANSWER: (A)

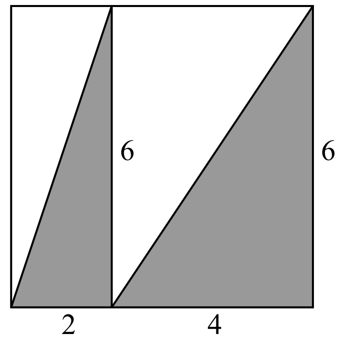
6. *Solution 1*
Consider the 6×6 square as being made up of a 2×6 rectangle on the left and a 4×6 rectangle on the right.



Each of these rectangles is divided in half by its diagonal, and so is half shaded.
Therefore, 50% of the total area is shaded.

Solution 2

The entire 6×6 square has area $6^2 = 36$.
Both shaded triangles have height 6 (the height of the square).
The bases of the triangles have lengths 2 and 4:



The left-hand triangle has area $\frac{1}{2} \times 2 \times 6 = 6$.
The right-hand triangle has area $\frac{1}{2} \times 4 \times 6 = 12$.
Therefore, the total shaded area is $6 + 12 = 18$, which is one-half (or 50%) of the area of the entire square.

Answer: (A)

7. There are $5 + 7 + 8 = 20$ ties in the box, 8 of which are pink.
When Stephen removes a tie at random, the probability of choosing a pink tie is $\frac{8}{20}$ which is equivalent to $\frac{2}{5}$.

Answer: (C)

8. The section of the number line between 0 and 5 has length $5 - 0 = 5$.
Since this section is divided into 20 equal parts, the width of each part is $\frac{5}{20} = \frac{1}{4} = 0.25$.
Since S is 5 of these equal parts to the right of 0, then $S = 0 + 5 \times 0.25 = 1.25$.
Since T is 5 of these equal parts to the left of 5, then $T = 5 - 5 \times 0.25 = 5 - 1.25 = 3.75$.
Therefore, $S + T = 1.25 + 3.75 = 5$.

Answer: (E)

9. If $\heartsuit = 1$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 1 \times 1 \times 1 = 1$, which is not possible since ∇ and $\heartsuit$ must be different positive integers.
If $\heartsuit = 2$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 2 \times 2 \times 2 = 8$, which is possible.
If $\heartsuit = 3$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 3 \times 3 \times 3 = 27$, which is not possible since ∇ is less than 20.
If $\heartsuit$ is greater than 3, then ∇ will be greater than 27 and so $\heartsuit$ cannot be greater than 3.
Thus, $\heartsuit = 2$ and so $\nabla = 8$.
This means that $\nabla \times \nabla = 8 \times 8 = 64$.

Answer: (D)

10. The line that passes through $(-2, 1)$ and $(2, 5)$ has slope $\dfrac{5-1}{2-(-2)} = \dfrac{4}{4} = 1$.
This means that, for every 1 unit to the right, the line moves 1 unit up.
Thus, moving 2 units to the right from the starting point $(-2, 1)$ to $x = 0$ will give a rise of 2.
Therefore, the line passes through $(-2 + 2, 1 + 2) = (0, 3)$.

Answer: (C)

11. The complete central angle of a circle measures 360°.
This means that the angle in the circle graph associated with Playing is $360^\circ - 130^\circ - 110^\circ$ or 120°.
A central angle of 120° represents $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the total central angle.
This means that the baby polar bear plays for $\frac{1}{3}$ of the day, which is $\frac{1}{3} \times 24 = 8$ hours.

Answer: (C)

12. Of the given uniform numbers,

- 11 and 13 are prime numbers
- 16 is a perfect square
- 12, 14 and 16 are even

Since Karl's and Liu's numbers were prime numbers, then their numbers were 11 and 13 in some order.
Since Glenda's number was a perfect square, then her number was 16.
Since Helga's and Julia's numbers were even, then their numbers were 12 and 14 in some order. (The number 16 is already taken.)
Thus, Ioana's number is the remaining number, which is 15.

Answer: (D)

13. Since the given equilateral triangle has side length 10, its perimeter is $3 \times 10 = 30$.
In terms of x, the perimeter of the given rectangle is $x + 2x + x + 2x = 6x$.
Since the two perimeters are equal, then $6x = 30$ which means that $x = 5$.
Since the rectangle is x by $2x$, its area is $x(2x) = 2x^2$.
Since $x = 5$, its area is $2(5^2) = 50$.

Answer: (B)

14. The average of the numbers $7, 9, 10, 11$ is $\frac{7+9+10+11}{4} = \frac{37}{4} = 9.25$, which is not equal to 18, which is the fifth number.
The average of the numbers $7, 9, 10, 18$ is $\frac{7+9+10+18}{4} = \frac{44}{4} = 11$, which is equal to 11, the remaining fifth number.
We can check that the averages of the remaining three combinations of four numbers is not equal to the fifth number.
Therefore, the answer is 11.
(We note that in fact the average of the original five numbers is $\frac{7+9+10+11+18}{5} = \frac{55}{5} = 11$, and when we remove a number that is the average of a set, the average does not change. Can you see why?)

Answer: (D)

15. We would like to find the first time after 4:56 where the digits are consecutive digits in increasing order.
It would make sense to try 5:67, but this is not a valid time.
Similarly, the time cannot start with 6, 7, 8 or 9.
No time starting with 10 or 11 starts with consecutive increasing digits.
Starting with 12, we obtain the time 12:34. This is the first such time.

We need to determine the length of time between 4:56 and 12:34.
From 4:56 to 11:56 is 7 hours, or $7 \times 60 = 420$ minutes.
From 11:56 to 12:00 is 4 minutes.
From 12:00 to 12:34 is 34 minutes.
Therefore, from 4:56 to 12:34 is $420 + 4 + 34 = 458$ minutes.

ANSWER: (A)

16. First, we note that we cannot have $n \leq 6$, since the first 6 letters are X's.
After 6 X's and 3 Y's, there are twice as many X's as Y's. In this case, $n = 6 + 3 = 9$.
After 6 X's and 12 Y's, there are twice as many Y's as X's. In this case, $n = 6 + 12 = 18$.
The next letters are all Y's (with 24 Y's in total), so there are no additional values of n with $n \leq 6 + 24 = 30$.
At this point, there are 6 X's and 24 Y's.
After 24 Y's and 12 X's (that is, 6 additional X's), there are twice as many Y's as X's. In this case, $n = 24 + 12 = 36$.
After 24 Y's and 48 X's (that is, 42 additional X's), there are twice as many X's as Y's. In this case, $n = 24 + 48 = 72$.
Since we are told that there are four values of n, then we have found them all, and their sum is $9 + 18 + 36 + 72 = 135$.

ANSWER: (C)

17. We note that $n = p^2q^2 = (pq)^2$.
Since $n < 1000$, then $(pq)^2 < 1000$ and so $pq < \sqrt{1000} \approx 31.6$.
Finding the number of possible values of n is thus equivalent to finding the number of positive integers m with $1 \leq m \leq 31 < \sqrt{1000}$ that are the product of two prime numbers.
The prime numbers that are at most 31 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.
The distinct products of pairs of these that are at most 31 are:

$$2 \times 3 = 6 \quad 2 \times 5 = 10 \quad 2 \times 7 = 14 \quad 2 \times 11 = 22 \quad 2 \times 13 = 26$$

$$3 \times 5 = 15 \quad 3 \times 7 = 21$$

Any other product either duplicates one that we have counted already, or is larger than 31.
Therefore, there are 7 such values of n.

ANSWER: (E)

18. Consider $\triangle PQR$.
Since the sum of the angles in a triangle is 180°, then

$$\angle QPR + \angle QRP = 180^\circ - \angle PQR = 180^\circ - 120^\circ = 60^\circ$$

Since $\angle QPS = \angle RPS$, then $\angle RPS = \frac{1}{2}\angle QPR$.
Since $\angle QRS = \angle PRS$, then $\angle PRS = \frac{1}{2}\angle QRP$.
Therefore,

$$\begin{aligned} \angle RPS + \angle PRS &= \tfrac{1}{2}\angle QPR + \tfrac{1}{2}\angle QRP \\ &= \tfrac{1}{2}(\angle QPR + \angle QRP) \\ &= \tfrac{1}{2} \times 60^\circ \\ &= 30^\circ \end{aligned}$$

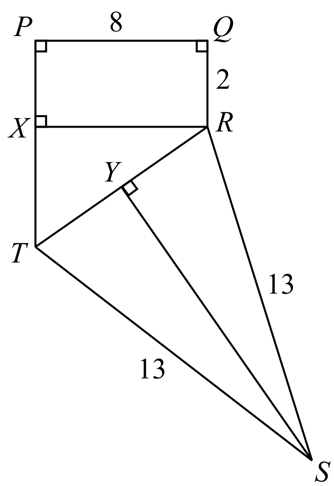
Finally, $\angle PSR = 180^\circ - (\angle RPS + \angle PRS) = 180^\circ - 30^\circ = 150^\circ$.

ANSWER: (E)

19. We recall that time $= \dfrac{\text{distance}}{\text{speed}}$. Travelling x km at 90 km/h takes $\dfrac{x}{90}$ hours.
Travelling x km at 120 km/h takes $\dfrac{x}{120}$ hours.
We are told that the difference between these lengths of time is 16 minutes.
Since there are 60 minutes in an hour, then 16 minutes is equivalent to $\dfrac{16}{60}$ hours.
Since the time at 120 km/h is 16 minutes less than the time at 90 km/h, then $\dfrac{x}{90} - \dfrac{x}{120} = \dfrac{16}{60}$.
Combining the fractions on the left side using a common denominator of $360 = 4 \times 90 = 3 \times 120$, we obtain $\dfrac{x}{90} - \dfrac{x}{120} = \dfrac{4x}{360} - \dfrac{3x}{360} = \dfrac{x}{360}$.
Thus, $\dfrac{x}{360} = \dfrac{16}{60}$.
Since $360 = 6 \times 60$, then $\dfrac{16}{60} = \dfrac{16 \times 6}{360} = \dfrac{96}{360}$. Thus, $\dfrac{x}{360} = \dfrac{96}{360}$ which means that $x = 96$.

Answer: (D)

20. We drop a perpendicular from R to X on PT, join R to T, and drop a perpendicular from S to Y on RT.



Since quadrilateral $PQRX$ has three right angles (at P, Q and X), then it must have four right angles and so is a rectangle. Its area is $8 \times 2 = 16$.
Next, $\triangle RXT$ is right-angled at X.
Since $PQRX$ is a rectangle, then $XR = PQ = 8$ and $PX = QR = 2$.
Since $PX = 2$, then $XT = PT - PX = 8 - 2 = 6$.
Thus, the area of $\triangle RXT$ is $\frac{1}{2} \times XT \times XR = \frac{1}{2} \times 6 \times 8 = 24$.
By the Pythagorean Theorem, $TR = \sqrt{XT^2 + XR^2} = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$, since $TR > 0$.
Since $\triangle TSR$ is isosceles with $ST = SR$ and SY is perpendicular to TR, then Y is the midpoint of TR.
Since $TY = YR = \frac{1}{2}TR$, then $TY = YR = 5$.
By the Pythagorean Theorem, $SY = \sqrt{ST^2 - TY^2} = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$, since $SY > 0$.
Therefore, the area of $\triangle STR$ is $\frac{1}{2} \times TR \times SY = \frac{1}{2} \times 10 \times 12 = 60$.
Finally, the area of pentagon $PQRST$ is the sum of the areas of the pieces, or $60 + 24 + 16 = 100$.

Answer: (D)

21. We determine the number of ways to get to each unshaded square in the grid obeying the given rules.
In the first row, there is 1 way to get to each unshaded square: by starting at that square.
In each row below the first, the number of ways to get to an unshaded square equals the sum of the number of ways to get to each of the unshaded squares diagonally up and to the left and up and to the right from the given square. This is because any path passing through any unshaded square needs to come from exactly one of these unshaded squares in the row above.
In the second row, there are 2 ways to get to each unshaded square: 1 way from each of two squares in the row above.
In the third row, there are 2 ways to get to each of the outside unshaded squares and 4 ways to get to the middle unshaded square.
Continuing in this way, we obtain the following number of ways to get to each unshaded square:

1		1		1
	2		2	
2		4		2
	6		6	
6		12		6

Since there are 6, 12 and 6 ways to get to the unshaded squares in the bottom row, then there are $6 + 12 + 6 = 24$ paths through the grid that obey the given rules.

ANSWER: (D)

22. Each wire has 2 ends.
Thus, 13 788 wires have $13\,788 \times 2 = 27\,576$ ends.
In a Miniou circuit, there are 3 wires connected to each node.
This means that 3 wire ends arrive at each node, and so there are $27\,576 \div 3 = 9192$ nodes.

ANSWER: (B)

23. The circle with centre P has radius 1 and passes through Q.
This means that $PQ = 1$.
Therefore, the circle with diameter PQ has radius $\frac{1}{2}$ and so has area $\pi\left(\frac{1}{2}\right)^2 = \frac{1}{4}\pi$.
To find the area of the shaded region, we calculate the area of the region common to both circles and subtract the area of the circle with diameter PQ.
Suppose that the two circles intersect at X and Y.
Join X to Y, P to Q, P to X, P to Y, Q to X, and Q to Y (Figure 1).
By symmetry, the area of the shaded region on each side of XY will be the same.
The area of the shaded region on the right side of XY equals the area of sector $PXQY$ of the left circle minus the area of $\triangle PXY$ (Figure 2).

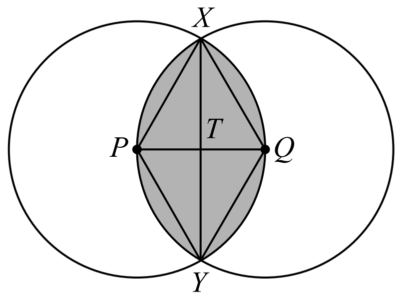


Figure 1

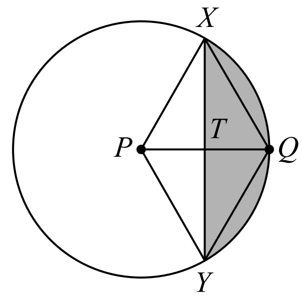


Figure 2

Since each of the large circles has radius 1, then $PQ = PX = PY = QX = QY = 1$.
This means that each of $\triangle XPQ$ and $\triangle YPQ$ is equilateral, and so $\angle XPQ = \angle YPQ = 60^\circ$.
Therefore, $\angle XPY = 120^\circ$, which means that sector $PXQY$ is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the full circle, and so has area $\frac{1}{3}\pi 1^2 = \frac{1}{3}\pi$.
Lastly, consider $\triangle PXY$.
Note that $PX = PY = 1$ and that $\angle XPQ = \angle YPQ = 60^\circ$.
Since $\triangle PXY$ is isosceles and PQ bisects $\angle XPY$, then PQ is perpendicular to XY at T and $XT = TY$.
By symmetry, $PT = TQ$. Since $PQ = 1$, then $PT = \frac{1}{2}$.
By the Pythagorean Theorem in $\triangle PTX$ (which is right-angled at T),

$$XT = \sqrt{PX^2 - PT^2} = \sqrt{1^2 - \left(\tfrac{1}{2}\right)^2} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2}$$

since $XT > 0$.
Therefore, $XY = 2XT = \sqrt{3}$.
The area of $\triangle PXY$ equals $\frac{1}{2}(XY)(PT) = \frac{1}{2}(\sqrt{3})\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4}$.
Now, we can calculate the area of the shaded region to the right of XY to be $\frac{1}{3}\pi - \frac{\sqrt{3}}{4}$, the difference between the area of sector $PXQY$ and the area of $\triangle PXT$.
Therefore, the area of the shaded region with the circle with diameter PQ removed is

$$2\left(\tfrac{1}{3}\pi - \tfrac{\sqrt{3}}{4}\right) - \tfrac{1}{4}\pi = \tfrac{2}{3}\pi - \tfrac{\sqrt{3}}{2} - \tfrac{1}{4}\pi = \tfrac{5}{12}\pi - \tfrac{\sqrt{3}}{2} \approx 0.443$$

Of the given choices, this is closest to 0.44.

ANSWER: (E)

24. We refer to the students with height 1.60 m as "taller" students and to those with height 1.22 m as "shorter" students.
For the average of four consecutive heights to be greater than 1.50 m, the sum of these four heights must be greater than $4 \times 1.50 \text{ m} = 6.00 \text{ m}$.
If there are 2 taller and 2 shorter students, then the sum of their heights is $2 \times 1.60 \text{ m} + 2 \times 1.22 \text{ m}$ or 5.64 m, which is not large enough.
Therefore, there must be more taller and fewer shorter students in a given group of 4 consecutive students.
If there are 3 taller students and 1 shorter student, then the sum of their heights is equal to $3 \times 1.60 \text{ m} + 1 \times 1.22 \text{ m}$ or 6.02 m, which is large enough.
Thus, in Mrs. Warner's line-up, any group of 4 consecutive students must include at least 3 taller students and at most 1 shorter student. (4 taller and 0 shorter students also give an average height that is greater than 1.50 m.)
For the average of seven consecutive heights to be less than 1.50 m, the sum of these seven heights must be less than $7 \times 1.50 \text{ m} = 10.50 \text{ m}$.
Note that 6 taller students and 1 shorter student have total height $6 \times 1.60 \text{ m} + 1 \times 1.22 \text{ m}$ or 10.82 m and that 5 taller students and 2 shorter students have total height equal to $5 \times 1.60 \text{ m} + 2 \times 1.22 \text{ m}$ or 10.44 m.
Thus, in Mrs. Warner's line-up, any group of 7 consecutive students must include at most 5 taller students and at least 2 shorter students.
Now, we determine the maximum possible length for a line-up. We use "T" to represent a taller student and "S" to represent a shorter student.
After some fiddling, we discover the line-up TTSTTTSTT.
The line-up TTSTTTSTT has length 9 and has the property that each group of 4 consecutive students includes exactly 3 T's and each group of 7 consecutive students includes exactly 5 Ts and so has the desired average height properties.
We claim that this is the longest such line-up, which makes the final answer 9 or (D).
Suppose that there was a line-up of length at least 10, and the first 10 heights in the line-up were $abcdefghjk$. Consider the following table of heights:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & h \\ c & d & e & f & g & h & j \\ d & e & f & g & h & j & k \end{array}$$

We explain why this table demonstrates that it is not possible to have at least 10 in the line-up.
Each row in this table is a list of 7 consecutive people from the line-up $abcdefghjk$ and so its sum is less than 10.50 m.
Each column in this table is a list of 4 consecutive people from the line-up $abcdefghjk$ and so its sum is greater than 6.00 m.
The sum of the numbers in this table equals the sum of the sums of the 4 rows, which must be less than 4×10.50 m or 42.00 m.
The sum of the numbers in this table equals the sum of the sums of the 7 columns, which must be greater than 7×6.00 m or 42.00 m.
The sum cannot be both less than and greater than 42.00 m.
This means that our assumption is incorrect, and so it must be impossible to have a line-up of length 10 or greater.
(There are a number of other ways to convince yourself that there cannot be more than 9 students in the line-up.)

ANSWER: (D)

25. Suppose that $m = 500$ and $1 \leq n \leq 499$ and $1 \leq r \leq 15$ and $2 \leq s \leq 9$ and $t = 0$.
Since $s > 0$, then the algorithm says that t is the remainder when r is divided by s.
Since $t = 0$, then r is a multiple of s. Thus, $r = as$ for some positive integer a.
Since $r > 0$, then the algorithm says that s is the remainder when n is divided by r.
In other words, $n = br + s$ for some positive integer b.
But $r = as$, so $n = bas + s = (ba + 1)s$.
In other words, n is a multiple of s, say $n = cs$ for some positive integer c.
Since $n > 0$, then r is the remainder when m is divided by n.
In other words, $m = dn + r$ for some positive integer d.
But $r = as$ and $n = cs$ so $m = dcs + as = (dc + a)s$.
In other words, m is a multiple of s, say $m = es$ for some positive integer e.
But $m = 500$ and $2 \leq s \leq 9$.
Since m is a multiple of s, then s is a divisor of 500 and so the possible values of s are $s = 2, 4, 5$.
(None of $1, 3, 6, 7, 8, 9$ is a divisor of 500.)
We know that r is a multiple of s, that $r > s$ (because s is the remainder when n is divided by r), and that $1 \leq r \leq 15$.
If $s = 5$, then $r = 10$ or $r = 15$.
If $s = 4$, then $r = 8$ or $r = 12$.
If $s = 2$, then $r = 4, 6, 8, 10, 12, 14$.

Suppose that $s = 5$ and $r = 10$.
Since $m = dn + r$, then $500 = dn + 10$ and so $dn = 490$.
Therefore, n is a divisor of 490, is a multiple of 5 (because $n = cs$), must be greater than $r = 10$, and must be 5 more than a multiple of 10 (because the remainder when n is divided by r is s).
Since $490 = 5 \times 2 \times 7^2$, then the divisors of 490 that are multiples of 5 are $5, 10, 35, 70, 245, 490$ (these are 5 times the divisors of 2×7^2). Among these, those greater than $r = 10$ having remainder 5 when divided by 10 are 35 and 245, and so the possible values of n in this case are 35 and 245.

For each possible pair s and r, we determine the values of n that satisfy the following conditions:

- n is a divsior of $500 - r$,
- n is a multiple of s,
- n is greater than r, and
- the remainder when n is divided by r is s.

We make a table:

s	r	$500 - r$	Divisors of $500 - r$ that are multiples of s	Possible n
5	10	$490 = 5 \times 2 \times 7^2$	$5, 10, 35, 70, 245, 490$	$35, 245$
5	15	$485 = 5 \times 97$	$5, 485$	485
4	8	$492 = 4 \times 3 \times 41$	$4, 12, 164, 492$	$12, 164, 492$
4	12	$488 = 4 \times 2 \times 61$	$4, 8, 244, 488$	244
2	4	$496 = 2 \times 2^3 \times 31$	$2, 4, 8, 16, 62, 124, 248, 496$	62
2	6	$494 = 2 \times 13 \times 19$	$2, 26, 38, 494$	$26, 38, 494$
2	8	$492 = 2 \times 2 \times 3 \times 41$	$2, 4, 6, 12, 82, 164, 246, 492$	82
2	10	$490 = 2 \times 5 \times 7^2$	$2, 10, 14, 70, 98, 490$	None
2	12	$488 = 2 \times 2^2 \times 61$	$2, 4, 8, 122, 244, 488$	122
2	14	$486 = 2 \times 3^5$	$2, 6, 18, 54, 162, 486$	None

In each row, we calculate the prime factorization in $500 - r$ in the third column, list the divisors of $500 - r$ that are multiples of s in the fourth column, and determine which of these are greater than r and give a remainder of s when divided by r in the fifth column.

Therefore, the possible values of n are $35, 245, 485, 12, 164, 492, 244, 62, 26, 38, 494, 82, 122$, of which there are 13.

Answer: (E)