The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Hypatia Contest

Thursday, April 12, 2018
(in North America and South America)

Friday, April 13, 2018
(outside of North America and South America)

Solutions

1. (a) The average of Aneesh's first six test scores was $\dfrac{17+13+20+12+18+10}{6} = \dfrac{90}{6} = 15$.

   (b) After Jon's third test, his average score was 14, and so the sum of the scores on his first three tests was $14 \times 3 = 42$.
   The sum of his scores on his first two tests was $17 + 12 = 29$, and so the score on his third test was $42 - 29 = 13$.
   (We may check that the average of $17, 12$ and 13 is $\dfrac{17+12+13}{3} = 14$.)

   (c) Dina wrote six tests followed by n more tests, for a total of $n + 6$ tests.
   After Dina's first 6 tests, her average score was 14, and so the sum of the scores on her first 6 tests was $14 \times 6 = 84$.
   Dina scored 20 on each of her next n tests, and so the sum of the scores on her next n tests was $20n$.
   Therefore, the sum of the scores on these $n + 6$ tests was $84 + 20n$.
   After Dina's $n + 6$ tests, her average score was 18, and so the sum of the scores on her $n + 6$ tests was $18(n + 6)$.
   Thus, $84 + 20n = 18(n + 6)$ or $84 + 20n = 18n + 108$ or $2n = 24$, and so $n = 12$.

2. (a) The distance from Botown to Aville is 120 km.
   Jessica drove this distance at a speed of 90 km/h, and so it took Jessica $\dfrac{120}{90} = \dfrac{4}{3}$ hours or $\dfrac{4}{3} \times 60 = 80$ minutes.

   (b) The distance from Botown to Aville is 120 km.
   The car predicted that Jessica would drive this distance at a speed of 80 km/h, and so it predicted that it would take Jessica $\dfrac{120}{80} = \dfrac{3}{2}$ hours or $\dfrac{3}{2} \times 60 = 90$ minutes.
   The ETA displayed by her car at 7:00 a.m. was 8:30 a.m..

   (c) Jessica drove from 7:00 a.m. to 7:16 a.m. (for 16 minutes) at a speed of 90 km/h, and so she travelled a distance of $\dfrac{16}{60} \times 90 = 24$ km.
   At 7:16 a.m., Jessica had a distance of 120 km $-$ 24 km $=$ 96 km left to travel.
   The car predicted that Jessica would drive this distance at a speed of 80 km/h, and so it predicted that it would take Jessica $\dfrac{96}{80} = \dfrac{6}{5}$ hours or $\dfrac{6}{5} \times 60 = 72$ minutes to complete the trip.
   The ETA displayed by her car at 7:16 a.m. was 72 minutes later or 8:28 a.m..

   (d) As in part (b), the car predicted that it would take Jessica 90 minutes or 1.5 hours to travel from Botown to Aville.
   Let the distance that Jessica travelled at 100 km/h be d km, and so the distance that Jessica travelled at 50 km/h was $(120 - d)$ km.
   The time that Jessica drove at 100 km/h was $\dfrac{d}{100}$ hours.
   The time that Jessica drove at 50 km/h was $\dfrac{120-d}{50}$ hours.
   Since the time predicted by her car is equal to the actual time that it took Jessica to travel from Botown to Aville, then $\dfrac{d}{100} + \dfrac{120-d}{50} = 1.5$.
   Solving for d, we get $d + 2(120 - d) = 1.5 \times 100$ or $-d + 240 = 150$, and so $d = 90$ km.
   Therefore, Jessica drove a distance of 90 km at a speed of 100 km/h.

3. (a) We are given that $T_1 = 1, T_2 = 2$ and $T_3 = 3$.
Evaluating, we get

$$T_4 = 1 + T_1T_2T_3 = 1 + (1)(2)(3) = 7, \text{ and}$$
$$T_5 = 1 + T_1T_2T_3T_4 = 1 + (1)(2)(3)(7) = 43.$$

(b) *Solution 1*
Each term after the second is equal to 1 more than the product of all previous terms in the sequence. Thus, $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$.
For all integers $n \geq 2$, we use the fact that $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$ to get

$$\begin{aligned} RS &= T_n^2 - T_n + 1 \\ &= T_n(T_n - 1) + 1 \\ &= T_n(1 + T_1T_2T_3 \cdots T_{n-1} - 1) + 1 \\ &= T_n(T_1T_2T_3 \cdots T_{n-1}) + 1 \\ &= T_1T_2T_3 \cdots T_{n-1}T_n + 1 \\ &= T_{n+1} \\ &= LS \end{aligned}$$

Solution 2
For all integers $n \geq 2$, we use the fact that $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$ to get

$$\begin{aligned} LS &= T_{n+1} \\ &= 1 + T_1T_2T_3 \cdots T_{n-1}T_n \\ &= 1 + (T_1T_2T_3 \cdots T_{n-1})T_n \\ &= 1 + (T_n - 1)T_n \\ &= T_n^2 - T_n + 1 \\ &= RS \end{aligned}$$

(c) Using the result from part (b), we get $T_n + T_{n+1} = T_n + T_n^2 - T_n + 1 = T_n^2 + 1$, for all integers $n \geq 2$.
Similarly,

$$\begin{aligned} T_nT_{n+1} - 1 &= T_n(T_n^2 - T_n + 1) - 1 \\ &= T_n^3 - T_n^2 + T_n - 1 \\ &= T_n^2(T_n - 1) + T_n - 1 \\ &= (T_n - 1)(T_n^2 + 1) \end{aligned}$$

Since $T_n + T_{n+1} = T_n^2 + 1$ and $T_n^2 + 1$ is a factor of $T_nT_{n+1} - 1$, then $T_n + T_{n+1}$ is a factor of $T_nT_{n+1} - 1$ for all integers $n \geq 2$.

(d) Using the result from part (b), we get $T_{2018} = T_{2017}^2 - T_{2017} + 1$.
Since T_{2017} is a positive integer greater than 1, then $T_{2017}^2 - T_{2017} + 1 > T_{2017}^2 - 2T_{2017} + 1$ and $T_{2017}^2 - T_{2017} + 1 < T_{2017}^2$.
That is, $T_{2017}^2 - 2T_{2017} + 1 < T_{2017}^2 - T_{2017} + 1 < T_{2017}^2$, and so $(T_{2017} - 1)^2 < T_{2018} < T_{2017}^2$.
Since $T_{2017} - 1$ and T_{2017} are two consecutive positive integers, then $(T_{2017} - 1)^2$ and T_{2017}^2 are two consecutive perfect squares, and so T_{2018} lies between two consecutive perfect squares and thus is not a perfect square.

4.(a)(i) By completing the square, the equations defining the two parabolas become

$$y = x^2 - 8x + 17 = x^2 - 8x + 16 + 1 = (x-4)^2 + 1, \text{ and}$$

$$y = -x^2 + 4x + 7 = -(x^2 - 4x + 4) + 11 = -(x-2)^2 + 11.$$

Thus, the parabola defined by the equation $y = x^2 - 8x + 17$ has vertex $V_1(4,1)$, and the parabola defined by the equation $y = -x^2 + 4x + 7$ has vertex $V_2(2,11)$.

(a)(ii) First, we determine the coordinates of the points of intersection P and Q.
When the two parabolas intersect,

$$\begin{aligned} x^2 - 8x + 17 &= -x^2 + 4x + 7 \\ 2x^2 - 12x + 10 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-5)(x-1) &= 0, \end{aligned}$$

and so the two parabolas intersect at $P(5,2)$ and $Q(1,10)$.
Next, we want to show why quadrilateral V_1PV_2Q is a parallelogram.
To do this, we will use the property that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
The midpoint of diagonal V_1V_2 is $\left(\dfrac{4+2}{2}, \dfrac{1+11}{2}\right)$ or $(3,6)$, and the midpoint of diagonal PQ is $\left(\dfrac{5+1}{2}, \dfrac{2+10}{2}\right)$ or $(3,6)$.
Since the midpoint of each diagonal is the same point, $(3,6)$, then the diagonals bisect each other and so quadrilateral V_1PV_2Q is a parallelogram.
(Note that we could have also shown that each pair of opposite sides of V_1PV_2Q is parallel.)

(b)(i) By completing the square, the equation defining the parabola $y = -x^2 + bx + c$ becomes

$$\begin{aligned} y &= -x^2 + bx + c \\ &= -(x^2 - bx) + c \\ &= -(x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4}) + c \\ &= -(x^2 - bx + \frac{b^2}{4}) + \frac{b^2}{4} + c \\ &= -(x - \frac{b}{2})^2 + \frac{b^2}{4} + c. \end{aligned}$$

The vertex of this parabola is $V_3\left(\dfrac{b}{2}, \dfrac{b^2}{4} + c\right)$ and the vertex of the parabola defined by the equation $y = x^2$ is $V_4(0,0)$.

First, we determine the conditions on b and c so that the points of intersection R and S exist and are distinct from one another.
When the two parabolas intersect, $-x^2 + bx + c = x^2$ or $2x^2 - bx - c = 0$.
This equation has two distinct real roots when its discriminant is greater than 0, or when $b^2 - 4(2)(-c) > 0$.
The points of intersection, R and S, exist and are distinct from one another when $c > \dfrac{-b^2}{8}$.
Next, we determine conditions on b and c so that each of R and S are distinct from both

V_3 and V_4.
The roots of the equation $2x^2 - bx - c = 0$ are given by the quadratic formula, and so $x = \dfrac{b \pm \sqrt{b^2+8c}}{4}$.
We let the x-coordinate of R be $x_1 = \dfrac{b+\sqrt{b^2+8c}}{4}$ and the x-coordinate of S be $x_2 = \dfrac{b-\sqrt{b^2+8c}}{4}$.
Each of the points R and S is not distinct from V_4 when $\dfrac{b \pm \sqrt{b^2+8c}}{4} = 0$ or $b = \mp\sqrt{b^2+8c}$ or $b^2 = b^2 + 8c$, and so $c = 0$.
Thus, we require that $c \neq 0$.
Similarly, each of the points R and S is not distinct from V_3 when $\dfrac{b \pm \sqrt{b^2+8c}}{4} = \dfrac{b}{2}$ or $b \pm \sqrt{b^2+8c} = 2b$ or $\pm\sqrt{b^2+8c} = b$ or $b^2 + 8c = b^2$, and so $c = 0$.
As before, we require that $c \neq 0$.
(Note that since R and V_4 lie on the same parabola, then if their x-coordinates are not equal, then they are distinct points – that is, we need not consider their y-coordinates. The same is true for points S and V_4, R and V_3, and S and V_3.)

Finally, we require that the vertices of the parabolas, $V_3\left(\dfrac{b}{2}, \dfrac{b^2}{4} + c\right)$ and $V_4(0,0)$, be distinct from one another.
Vertices V_3 and V_4 are distinct provided that if their x-coordinates are equal, then their y-coordinates are not equal (V_3 and V_4 lie on different parabolas and so we must consider both x- and y-coordinates).
If $\dfrac{b}{2} = 0$ or $b = 0$, then $\dfrac{b^2}{4} + c = \dfrac{0^2}{4} + c = c$, and since we have the requirement (from earlier) that $c \neq 0$, then vertices V_3 and V_4 are certainly distinct when $c \neq 0$.

If the two conditions $c > \dfrac{-b^2}{8}$ and $c \neq 0$ are satisfied, then for all pairs (b, c), the points R and S exist, and the points V_3, V_4, R, S are distinct.

(b)(ii) We begin by assuming that the conditions on b and c from part (b)(i) above are satisfied.
Thus, the points R and S exist, and the points V_3, V_4, R, S are distinct.
For quadrilateral V_3RV_4S to be a rectangle, it is sufficient to require that it be a parallelogram that has at least one pair of adjacent sides that are perpendicular to each other.
From (b)(i), the parabolas intersect at $R(x_1, x_1^2)$ and $S(x_2, x_2^2)$ (R and S each lie on the parabola $y = x^2$, and thus the y-coordinates are x_1^2 and x_2^2, respectively).
Recall that x_1 and x_2 are the distinct real roots of the quadratic equation $2x^2 - bx - c = 0$.
The sum of the roots of the general quadratic equation $Ax^2 + Bx + C = 0$ is equal to $\dfrac{-B}{A}$, and so $x_1 + x_2 = \dfrac{b}{2}$.
The product of the roots of the general quadratic equation $Ax^2 + Bx + C = 0$ is equal to $\dfrac{C}{A}$, and so $x_1 x_2 = \dfrac{-c}{2}$.

First, we will show that quadrilateral V_3RV_4S is a parallelogram since its diagonals bisect each other.

The midpoint of diagonal V_3V_4 is $\left(\dfrac{\frac{b}{2}+0}{2}, \dfrac{\frac{b^2}{4}+c+0}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2}{8}+\dfrac{c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2+4c}{8}\right)$.
The midpoint of diagonal RS is $\left(\dfrac{x_1+x_2}{2}, \dfrac{x_1^2+x_2^2}{2}\right)$.
However, $x_1+x_2=\dfrac{b}{2}$ and $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=\left(\dfrac{b}{2}\right)^2-2\left(\dfrac{-c}{2}\right)$, and so the midpoint of RS is $\left(\dfrac{\frac{b}{2}}{2}, \dfrac{\left(\frac{b}{2}\right)^2+c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2}{8}+\dfrac{c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2+4c}{8}\right)$.
Since the midpoint of diagonal V_3V_4 is equal to the midpoint of diagonal RS, then the diagonals bisect each other, and so V_3RV_4S is a parallelogram.

Next, we require that any one pair of adjacent sides of quadrilateral V_3RV_4S be perpendicular to each other. (This will mean that all pairs of adjacent sides are perpendicular.)
The slope of V_4S is $\dfrac{x_2^2-0}{x_2-0}=x_2$ since $x_2\neq 0$ ($S(x_2,x_2^2)$ and $V_4(0,0)$ are distinct points).
Similarly, the slope of V_4R is $\dfrac{x_1^2-0}{x_1-0}=x_1$ since $x_1\neq 0$ ($R(x_1,x_1^2)$ and $V_4(0,0)$ are distinct points).
Sides V_4S and V_4R are perpendicular to each other if the product of their slopes, x_1x_2, is equal to -1.
Since $x_1x_2=\dfrac{-c}{2}$, then $\dfrac{-c}{2}=-1$, and so $c=2$.

In addition to the condition that $c=2$, the two conditions from part (b)(i), $c>\dfrac{-b^2}{8}$ and $c\neq 0$, must also be satisfied.
Clearly if $c=2$, then $c\neq 0$.
Further, when $c=2$, $c>\dfrac{-b^2}{8}$ becomes $2>\dfrac{-b^2}{8}$ or $b^2>-16$ which is true for all real values of b.
The points R and S exist, the points V_3, V_4, R, S are distinct, and quadrilateral V_3RV_4S is a rectangle for all pairs (b,c) where $c=2$ and b is any real number.