The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Hypatia Contest

(Grade 11)

Thursday, April 12, 2018
(in North America and South America)

Friday, April 13, 2018
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes

Do not open this booklet until instructed to do so.

Number of questions: 4 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 
   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by 
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked  , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked  , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

1. Mr. Singh gives his students a test each week.

(a) Aneesh's scores on the first six tests were 17, 13, 20, 12, 18, and 10. What was the average (mean) of his test scores?

(b) Jon scored 17 and 12 on his first two tests. After the third test, his average (mean) score was 14. What was his score on the third test?

(c) After the first six tests, Dina had an average (mean) test score of 14. On each of the next n tests, Dina's score was 20 out of 20. After all of these tests, her average (mean) test score was 18. Determine the value of n.

2. Each day, Jessica drives from Botown to Aville, a distance of 120 km. During the drive, her car's navigation system constantly updates the estimated time of arrival (ETA) at Aville. The car predicts the ETA by assuming that Jessica will drive the remaining distance at 80 km/h.

(a) On Monday, Jessica drove at 90 km/h. How many minutes did it take Jessica to drive from Botown to Aville?

(b) On Tuesday, Jessica left Botown at 7:00 a.m.. What was the ETA displayed by her car at 7:00 a.m.?

(c) On Tuesday, Jessica drove at 90 km/h. Determine the ETA displayed by her car at 7:16 a.m..

(d) On Wednesday, Jessica noted the ETA predicted by her car when she left Botown. She travelled the first part of the trip at 50 km/h and travelled the rest of the way at 100 km/h. Jessica arrived in Aville at the ETA predicted by her car when she left Botown. Determine the distance that she drove at a speed of 100 km/h.

3. A sequence $T_1, T_2, T_3, \ldots$ is defined by $T_1 = 1$, $T_2 = 2$, and each term after the second is equal to 1 more than the product of all previous terms in the sequence. That is, $T_{n+1} = 1 + T_1 T_2 T_3 \cdots T_n$ for all integers $n \geq 2$. For example, $T_3 = 1 + T_1 T_2 = 3$.

(a) What is the value of T_5?

(b) Prove that $T_{n+1} = T_n^2 - T_n + 1$ for all integers $n \geq 2$.

(c) Prove that $T_n + T_{n+1}$ is a factor of $T_n T_{n+1} - 1$ for all integers $n \geq 2$.

(d) Prove that T_{2018} is not a perfect square.

4. 

(a) Consider the two parabolas defined by the equations $y = x^2 - 8x + 17$ and $y = -x^2 + 4x + 7$.

(i) Determine the coordinates of the vertices V_1 and V_2 of these two parabolas.

(ii) Suppose that these two parabolas intersect at the points P and Q. Explain why the quadrilateral $V_1 P V_2 Q$ is a parallelogram.

(b) The two parabolas defined by the equations $y = -x^2 + bx + c$ and $y = x^2$ have vertices V_3 and V_4, respectively. For some values of b and c, these parabolas intersect at the points R and S.

(i) Determine all pairs (b, c) for which the points R and S exist and the points V_3, V_4, R, S are distinct.

(ii) Determine all pairs (b, c) for which the points R and S exist, the points V_3, V_4, R, S are distinct, and quadrilateral $V_3 R V_4 S$ is a rectangle.

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For students...

Thank you for writing the 2018 Hypatia Contest! Each year, more than 240 000 students from more than 75 countries register to write the CEMC's Contests.

Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2018.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2018/2019 contests
- Register your students for the Canadian Senior and Intermediate Mathematics Contests which will be written in November
- Look at our free online courseware for senior high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results