The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2018 Fermat Contest

(Grade 11)

Tuesday, February 27, 2018
(in North America and South America)

Wednesday, February 28, 2018
(outside of North America and South America)

Solutions

1. Evaluating,

$$\begin{aligned}2016 - 2017 + 2018 - 2019 + 2020 &= 2016 + (2018 - 2017) + (2020 - 2019)\\ &= 2016 + 1 + 1\\ &= 2018\end{aligned}$$

Answer: (D)

2. Since the maximum temperature was 14°C and the minimum temperature was -11°C, then the range of temperatures was $14^\circ\text{C} - (-11^\circ\text{C}) = 25^\circ$C.

Answer: (B)

3. The expression $(3x + 2y) - (3x - 2y)$ is equal to $3x + 2y - 3x + 2y$ which equals $4y$.
When $x = -2$ and $y = -1$, this equals $4(-1)$ or -4.

Answer: (A)

4. The fraction $\frac{5}{7}$ is between 0 and 1.
The fraction $\frac{28}{3}$ is equivalent to $9\frac{1}{3}$ and so is between 9 and 10.
Therefore, the integers between these two fractions are $1, 2, 3, 4, 5, 6, 7, 8, 9$, of which there are 9.

Answer: (B)

5. If $\heartsuit = 1$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 1 \times 1 \times 1 = 1$, which is not possible since ∇ and $\heartsuit$ must be different positive integers.
If $\heartsuit = 2$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 2 \times 2 \times 2 = 8$, which is possible.
If $\heartsuit = 3$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 3 \times 3 \times 3 = 27$, which is not possible since ∇ is less than 20.
If $\heartsuit$ is greater than 3, then ∇ will be greater than 27 and so $\heartsuit$ cannot be greater than 3.
Thus, $\heartsuit = 2$ and so $\nabla = 8$.
This means that $\nabla \times \nabla = 8 \times 8 = 64$.

Answer: (D)

6. Since $\angle QRT = 158^\circ$, then $\angle QRP = 180^\circ - \angle QRT = 180^\circ - 158^\circ = 22^\circ$.
Since $\angle PRS = \angle QRS$ and $\angle QRP = \angle PRS + \angle QRS$, then $\angle QRS = \frac{1}{2}\angle QRP = \frac{1}{2}(22^\circ) = 11^\circ$.
Since $\triangle QSR$ is right-angled at Q, then $\angle QSR = 180^\circ - 90^\circ - \angle QRS = 90^\circ - 11^\circ = 79^\circ$.

Answer: (E)

7. Since Bev has driven 312 km and still has 858 km left to drive, the distance from Waterloo to Marathon is $312\text{ km} + 858\text{ km} = 1170$ km.
The halfway point of the drive is $\frac{1}{2}(1170\text{ km}) = 585$ km from Waterloo.
To reach this point, she still needs to drive $585\text{ km} - 312\text{ km} = 273$ km.

Answer: (B)

8. A line segment joining two points is parallel to the x-axis exactly when the y-coordinates of the two points are equal.
Here, this means that $2k + 1 = 4k - 5$ and so $6 = 2k$ or $k = 3$.
(We can check that when $k = 3$, the coordinates of the points are $(3, 7)$ and $(8, 7)$.)

Answer: (A)

9. Since the area of rectangle $PQRS$ is 180 and $SR = 15$, then $PS = \dfrac{180}{15} = 12$.
Since $PS = 12$ and $US = 4$, then $PU = PS - US = 12 - 4 = 8$.
Since $\triangle PUT$ is right-angled at U, then by the Pythagorean Theorem,

$$TU = \sqrt{PT^2 - PU^2} = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$

since $TU > 0$.
In $\triangle PTS$, we can consider base PS and height TU.
Therefore, its area is $\frac{1}{2}(PS)(TU) = \frac{1}{2}(12)(6) = 36$.

Answer: (B)

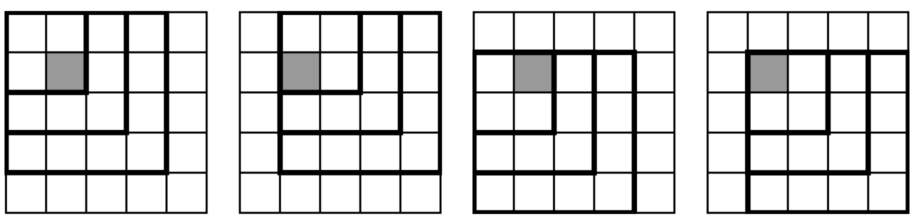
10. For any real number x not equal to 0, $x^2 > 0$.
Since $-1 < x < 0$, then $x^2 < (-1)^2 = 1$, and so $0 < x^2 < 1$.
Of the given points, only C is between 0 and 1.

Answer: (C)

11. Since $\frac{5}{6}$ of the balls are white and the remainder of the balls are red, then $\frac{1}{6}$ of the balls are red.
Since the 8 red balls represent $\frac{1}{6}$ of the total number of balls and $\frac{5}{6} = 5 \cdot \frac{1}{6}$, then the number of white balls is $5 \cdot 8 = 40$.

Answer: (C)

12. There is 1 square that is 1×1 that contains the shaded square (namely, the square itself).
There are 4 squares of each of the sizes 2×2, 3×3 and 4×4 that contain the shaded square.



Finally, there is 1 square that is 5×5 that contains the shaded square (namely, the 5×5 grid itself).
In total, there are thus $1 + 4 + 4 + 4 + 1 = 14$ squares that contain the shaded 1×1 square.

Answer: (E)

13. We would like to find the first time after 4:56 where the digits are consecutive digits in increasing order.
It would make sense to try 5:67, but this is not a valid time.
Similarly, the time cannot start with 6, 7, 8 or 9.
No time starting with 10 or 11 starts with consecutive increasing digits.
Starting with 12, we obtain the time 12:34. This is the first such time.
We need to determine the length of time between 4:56 and 12:34.
From 4:56 to 11:56 is 7 hours, or $7 \times 60 = 420$ minutes.
From 11:56 to 12:00 is 4 minutes.
From 12:00 to 12:34 is 34 minutes.
Therefore, from 4:56 to 12:34 is $420 + 4 + 34 = 458$ minutes.

Answer: (A)

14. The line with equation $y = x$ has slope 1 and passes through $(0, 0)$.
When this line is translated, its slope does not change.
When this line is translated 3 units to the right and 2 units down, every point on the line is translated 3 units to the right and 2 units down. Thus, the point $(0, 0)$ moves to $(3, -2)$.
Therefore, the new line has slope 1 and passes through $(3, -2)$.
Thus, its equation is $y - (-2) = 1(x - 3)$ or $y + 2 = x - 3$ or $y = x - 5$.
The y-intercept of this line is -5.

Answer: (C)

15. Each entry in the grid must be a divisor of the product of the numbers in its row and the product of the numbers in its column.

			56
			135
	N		48
21	108	160	

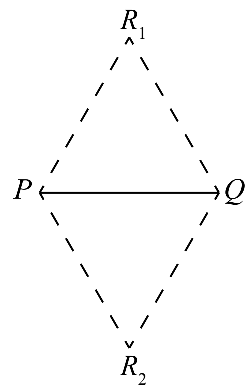
Only two of the products are multiples of 5, namely 160 and 135.
This means that the 5 must go in the second row and third column.
From this, we can see that the product of the other two numbers in the second row is $\frac{135}{5} = 27$.
Since all of the entries are between 1 and 9, then the remaining two numbers in this row must be 3 and 9.
Since 9 is not a divisor of 21, then 9 must be in the middle column.
This means that the product of the remaining numbers in the middle column is $\frac{108}{9} = 12$.
This means that the remaining digits in the middle column are 3 and 4, or 2 and 6. (These are the only factor pairs of 12 from the list of possible entries.)
Since 3 already occurs in the second row, then the entries in the second column must be 2 and 6.
Since 6 is not a divisor of 56, then 6 cannot go in the first row.
This means that 6 goes in the third row and so $N = 6$.
We can complete the grid as follows:

7	2	4	56
3	9	5	135
1	6	8	48
21	108	160	

Answer: (D)

16. *Solution 1*

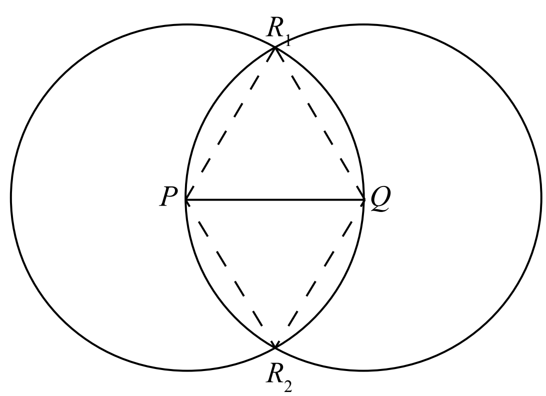
If point R is placed so that $PQ = QR = PR$, then the resulting $\triangle PQR$ is equilateral.
Since points P and Q are fixed, then there are two possible equilateral triangles with PQ as a side – one on each side of PQ.



One way to see this is to recognize that there are two possible lines through P that make an angle of 60° with PQ.

Solution 2

Consider the line segment PQ. Draw a circle with centre P that passes through Q and a circle with centre Q that passes through P.



Suppose that the point R satisfies $PQ = QR = PR$.
Since $PQ = QR$, then P and R are the same distance from Q, so R lies on the circle with centre Q that passes through P.
Since $PQ = PR$, then R lies on the circle with centre P that passes through Q.
In other words, point R is on both circles in the diagram.
Since these two circles intersect in exactly two points, then there are two possible locations for R.

Answer: (C)

17. The side length of the square is 2 and M and N are midpoints of sides.
Thus, $SM = MR = QN = NR = 1$.
Using the Pythagorean Theorem in $\triangle PSM$, we get $PM = \sqrt{PS^2 + SM^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$ since $PM > 0$.
Similarly, $PN = \sqrt{5}$.
Using the Pythagorean Theorem in $\triangle MNR$, we get $MN = \sqrt{MR^2 + NR^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$ since $MN > 0$.
Using the cosine law in $\triangle PMN$, we get

$$\begin{aligned} MN^2 &= PM^2 + PN^2 - 2(PM)(PN)\cos(\angle MPN) \\ 2 &= 5 + 5 - 2(\sqrt{5})(\sqrt{5})\cos(\angle MPN) \\ 2 &= 10 - 10\cos(\angle MPN) \\ 10\cos(\angle MPN) &= 8 \\ \cos(\angle MPN) &= \tfrac{8}{10} = \tfrac{4}{5} \end{aligned}$$

Answer: (A)

18. Suppose that $\sqrt{7 + \sqrt{48}} = m + \sqrt{n}$.
Squaring both sides, we obtain $7 + \sqrt{48} = (m + \sqrt{n})^2$.
Since $(m + \sqrt{n})^2 = m^2 + 2m\sqrt{n} + n$, then $7 + \sqrt{48} = (m^2 + n) + 2m\sqrt{n}$.
Let's make the assumption that $m^2 + n = 7$ and $2m\sqrt{n} = \sqrt{48}$.
Squaring both sides of the second equation, we obtain $4m^2n = 48$ or $m^2n = 12$.
So we have $m^2 + n = 7$ and $m^2n = 12$.
By inspection, we might see that $m = 2$ and $n = 3$ is a solution.
If we didn't see this by inspection, we could note that $n = 7 - m^2$ and so $m^2(7 - m^2) = 12$ or $m^4 - 7m^2 + 12 = 0$.
Factoring, we get $(m^2 - 3)(m^2 - 4) = 0$.
Since m is an integer, then $m^2 \neq 3$.
Thus, $m^2 = 4$ which gives $m = \pm 2$. Since m is a positive integer, then $m = 2$.
When $m = 2$, we get $n = 7 - m^2 = 3$.
Therefore, $m = 2$ and $n = 3$, which gives $m^2 + n^2 = 13$.
We note that $m + \sqrt{n} = 2 + \sqrt{3}$ and that $(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} = 7 + \sqrt{48}$, as required. This means that, while the assumption we made at the beginning was not fully general, it did give us an answer to the problem.

Answer: (E)

19. *Solution 1*
Over the first 3 minutes of the race, Peter ran 48 m farther than Radford. Here is why:

> We note that at a time of 0 minutes, Radford was at the 30 m mark.
> If Radford ran d m over these 3 minutes, then he will be at the $(d+30)$ m mark after 3 minutes.
> Since Peter is 18 m ahead of Radford after 3 minutes, then Peter is at the $(d+30+18)$ m mark.
> This means that, in 3 minutes, Peter ran $(d + 48)$ m which is 48 m farther than Radford's d m.

Since each runs at a constant speed, then Peter runs $\dfrac{48\text{ m}}{3\text{ min}} = 16$ m/min faster than Radford.
Since Peter finishes the race after 7 minutes, then Peter runs for another 4 minutes.

Over these 4 minutes, he runs (4 min) $\cdot$ (16 m/min) $= 64$ m farther than Radford.
After 3 minutes, Peter was 18 m ahead of Radford.
Therefore, after 7 minutes, Peter is 18 m + 64 m = 82 m farther ahead than Radford, and so Radford is 82 m from the finish line.

Solution 2
As in Solution 1, suppose that Radford ran d m over the first 3 minutes and so Peter runs $(d + 48)$ m over these first 3 minutes.
Since Peter's speed is constant, he runs $\frac{4}{3}(d + 48)$ m over the next 4 minutes.
Since Radford's speed is constant, he runs $\frac{4}{3}d$ over these next 4 minutes.
This means that Peter runs a total of $(d + 48)$ m $+ \frac{4}{3}(d + 48)$ m $= \frac{7}{3}(d + 48)$ m.
Also, Radford is $(30 + d + \frac{4}{3}d)$ m from the start after 7 minutes, since he had a 30 m head start.
Thus, Radford's distance from the finish line, in metres, is

$$\tfrac{7}{3}(d + 48) - (30 + d + \tfrac{4}{3}d) = \tfrac{7}{3}d + 112 - 30 - d - \tfrac{4}{3}d = 82$$

Answer: (D)

20. We count the positive integers x for which the product

$$(x - 2)(x - 4)(x - 6) \cdots (x - 2016)(x - 2018) \qquad (*)$$

equals 0 and is less than 0 separately.
The product $(*)$ equals 0 exactly when one of the factors equals 0.
This occurs exactly when x equals one of $2, 4, 6, \ldots, 2016, 2018$.
These are the even integers from 2 to 2018, inclusive, and there are $\frac{2018}{2} = 1009$ such integers.
The product $(*)$ is less than 0 exactly when none of its factors is 0 and an odd number of its factors are negative.
We note further that for every integer x we have

$$x - 2 > x - 4 > x - 6 > \cdots > x - 2016 > x - 2018$$

When $x = 1$, we have $x - 2 = -1$ and so all 1009 factors are negative, making $(*)$ negative.
When $x = 3$, we have $x - 2 = 1$, $x - 4 = -1$ and all of the other factors are negative, giving 1008 negative factors and so a positive product.
When $x = 5$, we have $x - 2 = 3$, $x - 4 = 1$ and $x - 6 = -1$ and all of the other factors are negative, giving 1007 negative factors and so a negative product.
This pattern continues giving a negative value for $(*)$ for $x = 1, 5, 9, 13, \ldots, 2013, 2017$.
There are $1 + \frac{2017-1}{4} = 505$ such values (starting at 1, these occur every 4 integers).
When $x \geq 2019$, each factor is positive and so $(*)$ is positive.
Therefore, there are $1009 + 505 = 1514$ positive integers x for which the product $(*)$ is less than or equal to 0.

We should further justify the pattern that we found above.
Suppose that $x = 4n + 1$ for $n = 0, 1, 2, \ldots, 504$. (These are the integers $1, 5, 9, 13, \ldots, 2017$.)
Then $(*)$ becomes

$$(4n - 1)(4n - 3)(4n - 5) \cdots (4n - 2015)(4n - 2017)$$

The $2k$th factor is $(n - (4k - 1))$ and so when $n = 4k$, this factor is positive and the next factor is negative.
In other words, when $n = 2k$, the first $2k$ of these factors are positive and the remaining factors

are negative.
In other words, when $n = 2k$, there is an even number of positive factors.
Since the total number of factors is 1009, which is odd, then the number of negative factors is odd and so the product is negative.
In a similar way, we can show that if $x = 4n + 3$ for $n = 0, 1, 2, \ldots, 503$ (these are the integers $3, 7, 11, \ldots, 2011, 2015$), then the product is positive.
This confirms that this pattern continues.

Answer: (C)

21. Substituting $n = 1$ into the equation $a_{n+1} = a_n + a_{n+2} - 1$ gives $a_2 = a_1 + a_3 - 1$.
Since $a_1 = x$ and $a_3 = y$, then $a_2 = x + y - 1$.
Rearranging the given equation, we obtain $a_{n+2} = a_{n+1} - a_n + 1$ for each $n \geq 1$.
Thus,

$$\begin{aligned}
a_4 &= a_3 - a_2 + 1 = y - (x + y - 1) + 1 = 2 - x \\
a_5 &= a_4 - a_3 + 1 = (2 - x) - y + 1 = 3 - x - y \\
a_6 &= a_5 - a_4 + 1 = (3 - x - y) - (2 - x) + 1 = 2 - y \\
a_7 &= a_6 - a_5 + 1 = (2 - y) - (3 - x - y) + 1 = x \\
a_8 &= a_7 - a_6 + 1 = x - (2 - y) + 1 = x + y - 1
\end{aligned}$$

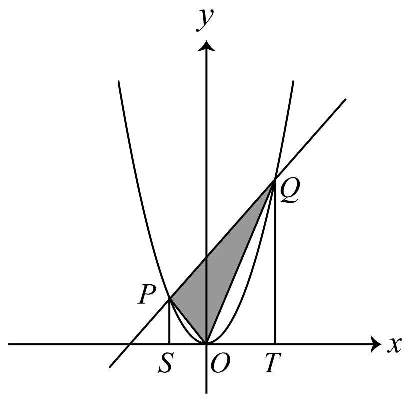
Since $a_7 = a_1$ and $a_8 = a_2$ and each term in the sequence depends only on the previous two terms, then the sequence repeats each 6 terms.
(For example, $a_9 = a_8 - a_7 + 1 = a_2 - a_1 + 1 = a_3$ and so on.)
Now

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = x + (x + y - 1) + y + (2 - x) + (3 - x - y) + (2 - y) = 6$$

which means that the sum of each successive group of 6 terms is also equal to 6.
We note that $2016 = 6 \cdot 336$ and so the 2016th term is the end of a group of 6 terms, which means that the sum of the first 2016 terms in the sequence is $6 \cdot 336 = 2016$.
Finally, $a_{2017} = a_1 = x$ and $a_{2018} = a_2 = x + y - 1$.
Thus, the sum of the first 2018 terms is $2016 + x + (x + y - 1) = 2x + y + 2015$.

Answer: (E)

22. First, we find the coordinates of the points P and Q in terms of k by finding the points of intersection of the graphs with equations $y = x^2$ and $y = 3kx + 4k^2$.
Equating values of y, we obtain $x^2 = 3kx + 4k^2$ or $x^2 - 3kx - 4k^2 = 0$.
We rewrite the left side as $x^2 - 4kx + kx + (-4k)(k) = 0$ which allows us to factor and obtain $(x - 4k)(x + k) = 0$ and so $x = 4k$ or $x = -k$.
Since $k > 0$, P is in the second quadrant and Q is in the first quadrant, then P has x-coordinate $-k$ (which is negative).
Since P lies on $y = x^2$, then its y-coordinate is $(-k)^2 = k^2$ and so the coordinates of P are $(-k, k^2)$.
Since Q lies on $y = x^2$ and has x-coordinate $4k$, then its y-coordinate is $(4k)^2 = 16k^2$ and so the coordinates of Q are $(4k, 16k^2)$.
Our next step is to determine the area of $\triangle OPQ$ in terms of k.
Since the area of $\triangle OPQ$ is numerically equal to 80, this will give us an equation for k which will allow us to find the slope of the line.
To find the area of $\triangle OPQ$ in terms of k, we drop perpendiculars from P and Q to S and T, respectively, on the x-axis.



The area of $\triangle OPQ$ is equal to the area of trapezoid $PSTQ$ minus the areas of $\triangle PSO$ and $\triangle QTO$.
Trapezoid $PSTQ$ has parallel bases SP and TQ and perpendicular height ST.
Since the coordinates of P are $(-k, k^2)$, then $SP = k^2$.
Since the coordinates of Q are $(4k, 16k^2)$, then $TQ = 16k^2$.
Also, $ST = 4k - (-k) = 5k$.
Thus, the area of trapezoid $PSTQ$ is $\frac{1}{2}(SP + TQ)(ST) = \frac{1}{2}(k^2 + 16k^2)(5k) = \frac{85}{2}k^3$.
$\triangle PSO$ is right-angled at S and so has area $\frac{1}{2}(SP)(SO) = \frac{1}{2}(k^2)(0 - (-k)) = \frac{1}{2}k^3$.
$\triangle QTO$ is right-angled at T and so has area $\frac{1}{2}(TQ)(TO) = \frac{1}{2}(16k^2)(4k - 0) = 32k^3$.
Combining these, the area of $\triangle POQ$ equals $\frac{85}{2}k^3 - \frac{1}{2}k^3 - 32k^3 = 10k^3$.
Since this area equals 80, then $10k^3 = 80$ or $k^3 = 8$ and so $k = 2$.
This means that the slope of the line is $3k$ which equals 6.

ANSWER: (D)

23. We are told that $(x - a)(x - 6) + 3 = (x + b)(x + c)$ for all real numbers x.
In particular, this equation holds when $x = 6$.
Substituting $x = 6$ gives $(6 - a)(6 - 6) + 3 = (6 + b)(6 + c)$ or $3 = (6 + b)(6 + c)$.
Since b and c are integers, then $6 + b$ and $6 + c$ are integers, which means that $6 + b$ is a divisor of 3.
Therefore, the possible values of $6 + b$ are $3, 1, -1, -3$.
These yield values for b of $-3, -5, -7, -9$.
We need to confirm that each of these values for b gives integer values for a and c.
If $b = -3$, then $6 + b = 3$. The equation $3 = (6 + b)(6 + c)$ tells us that $6 + c = 1$ and so $c = -5$.
When $b = -3$ and $c = -5$, the original equation becomes $(x - a)(x - 6) + 3 = (x - 3)(x - 5)$.
Expanding the right side gives $(x-a)(x-6)+3 = x^2-8x+15$ and so $(x-a)(x-6) = x^2-8x+12$.
The quadratic $x^2 - 8x + 12$ factors as $(x - 2)(x - 6)$ and so $a = 2$ and this equation is an identity that is true for all real numbers x.
Similarly, if $b = -5$, then $c = -3$ and $a = 2$. (This is because b and c are interchangeable in the original equation.)
Also, if $b = -7$, then $c = -9$ and we can check that $a = 10$.
Similarly, if $b = -9$, then $c = -7$ and $a = 10$.
Therefore, the possible values of b are $b = -3, -5, -7, -9$.
The sum of these values is $(-3) + (-5) + (-7) + (-9) = -24$.

ANSWER: (B)

24. We use the notation "$a/b/c$" to mean a pucks in one bucket, b pucks in a second bucket, and c pucks in the third bucket, ignoring the order of the buckets.

Yellow buckets

1/0/0: With 1 puck to distribute, the distribution will always be 1/0/0.

Blue buckets

Since there are 2 pucks to distribute amongst the three buckets, then there is a total of $3^2 = 9$ ways of doing this. (There are 3 possibilities for each of 2 pucks.)

2/0/0: There are 3 ways in which the 2 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{9}$.

1/1/0: Thus, there are $9 - 3 = 6$ ways in which the 2 pucks are distributed with 1 puck in each of two buckets and 0 pucks in the third bucket. The probability of this is $\frac{6}{9}$.

Red buckets

With 3 pucks to distribute amongst 3 buckets, there is a total of $3^3 = 27$ ways.

3/0/0: There are 3 ways in which the 3 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{27}$.

1/1/1: There are $3 \cdot 2 \cdot 1 = 6$ ways in which the 3 pucks end up with one in each bucket (3 choices of bucket for the first puck, 2 for the second, and 1 for the third). The probability of this is $\frac{6}{27}$.

2/1/0: Thus, there are $27 - 3 - 6 = 18$ ways in which the 2 pucks are distributed with 2 pucks in 1 bucket, 1 puck in 1 bucket, and 0 pucks in 1 bucket. The probability of this is $\frac{18}{27}$.

Green buckets

With 4 pucks to distribute amongst 3 buckets, there is a total of $3^4 = 81$ ways.

4/0/0: There are 3 ways in which the 4 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{81}$.

3/1/0: There are $4 \times 3 \times 2 = 24$ ways in which the pucks end up with 3 in one bucket and 1 in another (4 ways to choose a puck to be on its own, 3 ways to choose the bucket for this puck, and 2 ways to choose the bucket for the 3 pucks). The probability of this is $\frac{24}{81}$.

2/1/1: There are 6 ways of choosing two of the four pucks. (If they are labelled W, X, Y, Z, then we can choose WX, WY, WZ, XY, XZ, or YZ.) There are $6 \times 3 \times 2 = 36$ ways in which the pucks can be distributed with 2 pucks in one bucket and 1 puck in each of the remaining buckets (6 ways to choose the 2 pucks that go together, 3 ways to choose the bucket, and 2 ways in which the remaining 2 pucks can be assigned to the remaining 2 buckets). The probability of this is $\frac{36}{81}$.

2/2/0: Thus, there are $81 - 3 - 24 - 36 = 18$ ways in which the 4 pucks are distributed with 2 pucks in each of 2 buckets. The probability of this is $\frac{18}{81}$.

For a green bucket to contain more pucks than each of the other 11 buckets, the following possible distributions exist with probabilities as shown:

Green	Red	Blue	Yellow	Probability
4/0/0 ($p = \frac{3}{81}$)	Any ($p = 1$)	Any ($p = 1$)	Any ($p = 1$)	$\frac{3}{81}$
3/1/0 ($p = \frac{24}{81}$)	Any but 3/0/0 ($p = 1 - \frac{3}{27}$)	Any ($p = 1$)	Any ($p = 1$)	$\frac{24}{81} \cdot \frac{24}{27}$
2/1/1 ($p = \frac{36}{81}$)	1/1/1 ($p = \frac{6}{27}$)	1/1/0 ($p = \frac{6}{9}$)	Any ($p = 1$)	$\frac{36}{81} \cdot \frac{6}{27} \cdot \frac{6}{9}$

A 2/2/0 distribution of pucks among green buckets cannot satisfy the desired conditions because there would be not be a single green bucket with more pucks in it than any other bucket, as there would be two green buckets containing the same number of pucks.

Therefore, the overall probability is $\frac{3}{81} + \frac{24}{81} \cdot \frac{24}{27} + \frac{36}{81} \cdot \frac{6}{27} \cdot \frac{6}{9} = \frac{1}{27} + \frac{8}{27} \cdot \frac{8}{9} + \frac{4}{9} \cdot \frac{2}{9} \cdot \frac{2}{3} = \frac{9}{243} + \frac{64}{243} + \frac{16}{243} = \frac{89}{243}$.

ANSWER: (B)

25. Suppose that D is a digit and k is a positive integer. Then

$$D_{(k)} = \underbrace{DD\cdots DD}_{k\text{ times}} = D\cdot\underbrace{11\cdots 11}_{k\text{ times}} = D\cdot\tfrac{1}{9}\cdot\underbrace{99\cdots 99}_{k\text{ times}} = D\cdot\tfrac{1}{9}\cdot(1\underbrace{00\cdots 00}_{k\text{ times}}-1) = D\cdot\tfrac{1}{9}\cdot(10^k-1)$$

Therefore, the following equations are equivalent:

$$\begin{aligned}
P_{(2k)} - Q_{(k)} &= \left(R_{(k)}\right)^2 \\
P\cdot\tfrac{1}{9}\cdot(10^{2k}-1) - Q\cdot\tfrac{1}{9}\cdot(10^k-1) &= \left(R\cdot\tfrac{1}{9}\cdot(10^k-1)\right)^2 \\
P\cdot\tfrac{1}{9}\cdot(10^{2k}-1) - Q\cdot\tfrac{1}{9}\cdot(10^k-1) &= R^2\cdot\tfrac{1}{81}\cdot(10^k-1)^2 \\
9P\cdot(10^{2k}-1) - 9Q\cdot(10^k-1) &= R^2\cdot(10^k-1)^2 \\
9P\cdot(10^k-1)(10^k+1) - 9Q\cdot(10^k-1) &= R^2\cdot(10^k-1)^2 \\
9P\cdot(10^k+1) - 9Q &= R^2\cdot(10^k-1) \qquad (\text{since } 10^k-1\neq 0) \\
9P\cdot 10^k + 9K - 9Q &= R^2\cdot 10^k - R^2 \\
9P - 9Q + R^2 &= 10^k(R^2-9P)
\end{aligned}$$

We consider three cases: $3\leq k\leq 2018$, $k=1$, and $k=2$.

Case 1: $3\leq k\leq 2018$
Suppose that $R^2-9P\neq 0$.
Since $k\geq 3$, then $10^k(R^2-9P)>1000$ if $R^2-9P>0$ and $10^k(R^2-9P)<-1000$ if $R^2-9P<0$.
Since P,Q,R are digits, then $9P-9Q+R^2$ is at most $9(9)-9(0)+9^2=162$ and $9P-9Q+R^2$ is at least $9(0)-9(9)+0^2=-81$.
This means that if $R^2-9P\neq 0$, we cannot have $9P-9Q+R^2=10^k(R^2-9P)$ since the possible values do not overlap.
So if $3\leq k\leq 2018$, we must have $R^2-9P=0$ and so $9P-9Q+R^2=0$.
If $R^2=9P$, then R^2 is a multiple of 3 and so R is a multiple of 3.
Since R is a positive digit, then $R=3$ or $R=6$ or $R=9$.
If $R=3$, then $9P=R^2=9$ and so $P=1$.
Since $9P-9Q+R^2=0$, then $9Q=9(1)+9=18$ and so $Q=2$.
If $R=6$, then $9P=R^2=36$ and so $P=4$.
Since $9P-9Q+R^2=0$, then $9Q=9(4)+36=72$ and so $Q=8$.
If $R=9$, then $9P=R^2=81$ and so $P=9$.
Since $9P-9Q+R^2=0$, then $9Q=9(9)+81=162$ and so Q cannot be a digit.
Therefore, in the case where $3\leq k\leq 2018$, we obtain the quadruples $(P,Q,R,k)=(1,2,3,k)$ and $(P,Q,R,k)=(4,8,9,k)$.
Since there are $2018-3+1=2016$ possible values of k, then we have $2\cdot 2016=4032$ quadruples so far.

Case 2: $k=1$
Here, the equation $9P-9Q+R^2=10^k(R^2-9P)$ becomes $9P-9Q+R^2=10R^2-90P$ or $99P=9R^2+9Q$ or $11P=R^2+Q$.
For each possible value of P from 1 to 9, we determine the possible values of Q and R by looking for perfect squares that are at most 9 less than $11P$.
$P=1$: Here, $11P=11$ which is close to squares 4 and 9. We obtain $(R,Q)=(2,7),(3,2)$.
$P=2$: Here, $11P=22$ which is close to the square 16. We obtain $(R,Q)=(4,6)$.
$P=3$: Here, $11P=33$ which is close to the square 25. We obtain $(R,Q)=(5,8)$.
$P=4$: Here, $11P=44$ which is close to the square 36. We obtain $(R,Q)=(6,8)$.
$P=5$: Here, $11P=55$ which is close to the square 49. We obtain $(R,Q)=(7,6)$.

$P = 6$: Here, $11P = 66$ which is close to the square 64. We obtain $(R, Q) = (8, 2)$.
$P = 7$: There are no perfect squares between 68 and 76, inclusive.
$P = 8$: Here, $11P = 88$ which is close to the square 81. We obtain $(R, Q) = (9, 7)$.
$P = 9$: There are no perfect squares between 90 and 98, inclusive.
Since $k = 1$ in each of these cases, we obtain an additional 8 quadruples.

Case 3: $k = 2$
Here, the equation $9P - 9Q + R^2 = 10^k(R^2 - 9P)$ becomes $9P - 9Q + R^2 = 100R^2 - 900P$ or $909P = 99R^2 + 9Q$ or $101P = 11R^2 + Q$.
As P ranges from 1 to 9, the possible values of $101P$ are $101, 202, 303, 404, 505, 606, 707, 808, 909$.
As R ranges from 1 to 9, the possible values of $11R^2$ are $11, 44, 99, 176, 275, 396, 539, 704, 891$.
The pairs of integers in the first and second lists that differ by at most 9 are

(i) 101 and 99 (which give $(P, Q, R) = (1, 2, 3)$),
(ii) 404 and 396 (which give $(P, Q, R) = (4, 8, 6)$), and
(iii) 707 and 704 (which give $(P, Q, R) = (7, 3, 8)$).

Since $k = 2$ in each of these cases, we obtain an additional 3 quadruples.

In total, there are thus $N = 4032 + 8 + 3 = 4043$ quadruples.
The sum of the digits of N is $4 + 0 + 4 + 3 = 11$.

Answer: (C)