The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Euclid Contest

Wednesday, April 11, 2018
(in North America and South America)

Thursday, April 12, 2018
(outside of North America and South America)

Solutions

1. (a) When $x = 11$,

$$x + (x+1) + (x+2) + (x+3) = 4x + 6 = 4(11) + 6 = 50$$

Alternatively,

$$x + (x+1) + (x+2) + (x+3) = 11 + 12 + 13 + 14 = 50$$

(b) We multiply the equation $\frac{a}{6} + \frac{6}{18} = 1$ by 18 to obtain $3a + 6 = 18$.
Solving, we get $3a = 12$ and so $a = 4$.

(c) *Solution 1*
Since the cost of one chocolate bar is \$1.00 more than that of a pack of gum, then if we replace a pack of gum with a chocolate bar, then the price increases by \$1.00.
Starting with one chocolate bar and two packs of gum, we replace the two packs of gum with two chocolate bars.
This increases the price by \$2.00 from \$4.15 to \$6.15.
In other words, three chocolate bars cost \$6.15, and so one chocolate bar costs $\frac{1}{3}$(\$6.15) or \$2.05.

Solution 2
Let the cost of one chocolate bar be $\$x$.
Let the cost of one pack of gum be $\$y$.
Since the cost of one chocolate bar and two packs of gum is \$4.15, then $x + 2y = 4.15$.
Since one chocolate bar costs \$1.00 more than one pack of gum, then $x = y + 1$.
Since $x = y + 1$, then $y = x - 1$.
Since $x + 2y = 4.15$, then $x + 2(x - 1) = 4.15$.
Solving, we obtain $x + 2x - 2 = 4.15$ or $3x = 6.15$ and so $x = 2.05$.
In other words, the cost of one chocolate bar is \$2.05.

2. (a) Suppose that the five-digit integer has digits $abcde$.
The digits a, b, c, d, e are $1, 3, 5, 7, 9$ in some order.
Since $abcde$ is greater than 80 000, then $a \geq 8$, which means that $a = 9$.
Since $9bcde$ is less than 92 000, then $b < 2$, which means that $b = 1$.
Since $91cde$ has units (ones) digit 3, then $e = 3$.
So far, the integer is $91cd3$, which means that c and d are 5 and 7 in some order.
Since the two-digit integer cd is divisible by 5, then it must be 75.
This means that the the five-digit integer is 91753.

(b) By the Pythagorean Theorem in $\triangle ADB$,

$$AD^2 = AB^2 - BD^2 = 13^2 - 12^2 = 169 - 144 = 25$$

Since $AD > 0$, then $AD = \sqrt{25} = 5$.
By the Pythagorean Theorem in $\triangle CDB$,

$$CD^2 = BC^2 - BD^2 = (12\sqrt{2})^2 - 12^2 = 12^2(2) - 12^2 = 12^2$$

Since $CD > 0$, then $CD = 12$.
Therefore, $AC = AD + DC = 5 + 12 = 17$.

(c) *Solution 1*

The area of the shaded region equals the area of square $OABC$ minus the area of $\triangle OCD$.
Since square $OABC$ has side length 6, then its area is 6^2 or 36.
Also, $OC = 6$.
Since the equation of the line is $y = 2x$, then its slope is 2.
Since the slope of the line is 2, then $\dfrac{OC}{CD} = 2$.
Since $OC = 6$, then $CD = 3$.
Thus, the area of $\triangle OCD$ is $\frac{1}{2}(OC)(CD) = \frac{1}{2}(6)(3) = 9$.
Finally, the area of shaded region must be $36 - 9 = 27$.

Solution 2

Since square $OABC$ has side length 6, then $OA = AB = CB = OC = 6$.
Since the slope of the line is 2, then $\dfrac{OC}{CD} = 2$.
Since $OC = 6$, then $CD = 3$.
Since $CB = 6$ and $CD = 3$, then $DB = CB - CD = 3$.
The shaded region is a trapezoid with parallel sides $DB = 3$ and $OA = 6$ and height $AB = 6$.
Therefore, the area of the shaded region is $\frac{1}{2}(DB + OA)(AB) = \frac{1}{2}(3 + 6)(6) = 27$.

3. (a) Calculating, $\left(\sqrt{4+\sqrt{4}}\right)^4 = \left(\sqrt{4+2}\right)^4 = \left(\sqrt{6}\right)^4 = \left(\left(\sqrt{6}\right)^2\right)^2 = 6^2 = 36$.

(b) Since y is an integer, then $8 - y^2$ is an integer.
Therefore, $\sqrt{23 - x}$ is an integer which means that $23 - x$ is a perfect square.
Since x is a positive integer, then $23 - x < 23$ and so $23 - x$ must be a perfect square that is less than 23.
We make a table listing the possible values of $23 - x$ and the resulting values of x, $\sqrt{23 - x} = 8 - y^2$, y^2, and y:

$23 - x$	x	$\sqrt{23-x} = 8 - y^2$	y^2	y
16	7	4	4	± 2
9	14	3	5	$\pm\sqrt{5}$
4	19	2	6	$\pm\sqrt{6}$
1	22	1	7	$\pm\sqrt{7}$
0	23	0	8	$\pm\sqrt{8}$

Since x and y are positive integers, then we must have $(x, y) = (7, 2)$.
(We note that since we were told that there is only one such pair, we did not have to continue the table beyond the first row.)

(c) Since the line with equation $y = mx + 2$ passes through $(1, 5)$, then $5 = m + 2$ and so $m = 3$.
Since the parabola with equation $y = ax^2 + 5x - 2$ passes through $(1, 5)$, then $5 = a + 5 - 2$ and so $a = 2$.
To find the coordinates of Q, we determine the second point of intersection of $y = 3x + 2$ and $y = 2x^2 + 5x - 2$ by equating values of y:

$$\begin{aligned} 2x^2 + 5x - 2 &= 3x + 2 \\ 2x^2 + 2x - 4 &= 0 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \end{aligned}$$

Therefore, $x = 1$ or $x = -2$.
Since P has x-coordinate 1, then Q has x-coordinate -2.
Since Q lies on the line with equation $y = 3x + 2$, we have $y = 3(-2) + 2 = -4$.
In summary, (i) $m = 3$, (ii) $a = 2$, and (iii) the coordinates of Q are $(-2, -4)$.

4. (a) Since $80 = 2^4 \cdot 5$, its positive divisors are $1, 2, 4, 5, 8, 10, 16, 20, 40, 80$.
For an integer n to share exactly two positive common divisors with 80, these divisors must be either 1 and 2 or 1 and 5. (1 is a common divisor of any two integers. The second common divisor must be a prime number since any composite divisor will cause there to be at least one more common divisor which is prime.)
Since $1 \leq n \leq 30$ and n is a multiple of 2 or of 5, then the possible values of n come from the list

$$2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 25, 26, 28, 30$$

We remove the multiples of 4 from this list (since they would share at least the divisors $1, 2, 4$ with 80) and the multiples of 10 from this list (since they would share at least the divisors $1, 2, 5, 10$ with 80).
This leaves the list

$$2, 5, 6, 14, 15, 18, 22, 25, 26$$

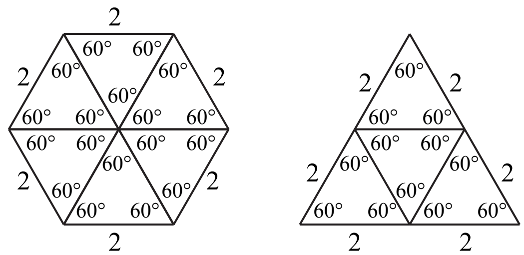
The common divisors of any number from this list and 80 are either 1 and 2 or 1 and 5.
There are 9 such integers.

(b) We start with $f(50)$ and apply the given rules for the function until we reach $f(1)$:

$$\begin{aligned} f(50) &= f(25) && \text{(since 50 is even and } \tfrac{1}{2}(50) = 25) \\ &= f(24) + 1 && \text{(since 25 is odd and } 25 - 1 = 24) \\ &= f(12) + 1 && (\tfrac{1}{2}(24) = 12) \\ &= f(6) + 1 && (\tfrac{1}{2}(12) = 6) \\ &= f(3) + 1 && (\tfrac{1}{2}(6) = 3) \\ &= (f(2) + 1) + 1 && (3 - 1 = 2) \\ &= f(1) + 1 + 1 && (\tfrac{1}{2}(2) = 1) \\ &= 1 + 1 + 1 && (f(1) = 1) \\ &= 3 \end{aligned}$$

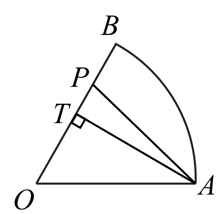
Therefore, $f(50) = 3$.

5. (a) Since the hexagon has perimeter 12 and has 6 sides, then each side has length 2.
Since equilateral $\triangle PQR$ has perimeter 12, then its side length is 4.
Consider equilateral triangles with side length 2.
Six of these triangles can be combined to form a regular hexagon with side length 2 and four of these can be combined to form an equilateral triangle with side length 4.



Note that the six equilateral triangles around the centre of the hexagon give a total central angle of $6 \cdot 60^\circ = 360^\circ$ (a complete circle) and the three equilateral triangles along each side of the large equilateral triangle make a straight angle of 180° (since $3 \cdot 60^\circ = 180^\circ$).
Also, the length of each side of the hexagon is 2 and the measure of each internal angle is 120°, which means that the hexagon is regular. Similarly, the triangle is equilateral.
Since the triangle is made from four identical smaller triangles and the hexagon is made from six of these smaller triangles, the ratio of the area of the triangle to the hexagon is $4 : 6$ which is equivalent to $2 : 3$.

(b) Since sector AOB is $\frac{1}{6}$ of a circle with radius 18, its area is $\frac{1}{6}(\pi \cdot 18^2)$ or 54π.
For the line AP to divide this sector into two pieces of equal area, each piece has area $\frac{1}{2}(54\pi)$ or 27π.
We determine the length of OP so that the area of $\triangle POA$ is 27π.
Since sector AOB is $\frac{1}{6}$ of a circle, then $\angle AOB = \frac{1}{6}(360^\circ) = 60^\circ$.
Drop a perpendicular from A to T on OB.



The area of $\triangle POA$ is $\frac{1}{2}(OP)(AT)$.
$\triangle AOT$ is a 30°-60°-90° triangle.
Since $AO = 18$, then $AT = \frac{\sqrt{3}}{2}(AO) = 9\sqrt{3}$.
For the area of $\triangle POA$ to equal 27π, we have $\frac{1}{2}(OP)(9\sqrt{3}) = 27\pi$ which gives $OP = \dfrac{54\pi}{9\sqrt{3}} = \dfrac{6\pi}{\sqrt{3}} = 2\sqrt{3}\pi$.
(Alternatively, we could have used the fact that the area of $\triangle POA$ is $\frac{1}{2}(OA)(OP)\sin(\angle POA)$.)

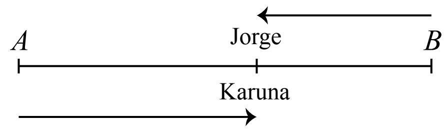
6. (a) Let $\theta = 10k^\circ$.

The given inequalities become $0^\circ < \theta < 180^\circ$ and $\dfrac{5\sin\theta - 2}{\sin^2\theta} \geq 2$.
When $0^\circ < \theta < 180^\circ$, $\sin\theta \neq 0$.
This means that we can can multiply both sides by $\sin^2\theta > 0$ and obtain the equivalent inequalities:

$$\begin{aligned} \frac{5\sin\theta - 2}{\sin^2\theta} &\geq 2 \\ 5\sin\theta - 2 &\geq 2\sin^2\theta \\ 0 &\geq 2\sin^2\theta - 5\sin\theta + 2 \\ 0 &\geq (2\sin\theta - 1)(\sin\theta - 2) \end{aligned}$$

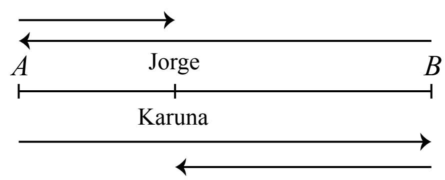
Since $\sin\theta \leq 1$, then $\sin\theta - 2 \leq -1 < 0$ for all θ.
Therefore, $(2\sin\theta - 1)(\sin\theta - 2) \leq 0$ exactly when $2\sin\theta - 1 \geq 0$.
Note that $2\sin\theta - 1 \geq 0$ exactly when $\sin\theta \geq \frac{1}{2}$.
Therefore, the original inequality is true exactly when $\frac{1}{2} \leq \sin\theta \leq 1$.
Note that $\sin 30^\circ = \sin 150^\circ = \frac{1}{2}$ and $0^\circ < \theta < 180^\circ$.
When $\theta = 0^\circ$, $\sin\theta = 0$.
From $\theta = 0^\circ$ to $\theta = 30^\circ$, $\sin\theta$ increases from 0 to $\frac{1}{2}$.
From $\theta = 30^\circ$ to $\theta = 150^\circ$, $\sin\theta$ increases from $\frac{1}{2}$ to 1 and then decreases to $\frac{1}{2}$.
From $\theta = 150^\circ$ to $\theta = 180^\circ$, $\sin\theta$ decreases from $\frac{1}{2}$ to 0.
Therefore, the original inequality is true exactly when $30^\circ \leq \theta \leq 150^\circ$ which is equivalent to $30^\circ \leq 10k^\circ \leq 150^\circ$ and to $3 \leq k \leq 15$.
The integers k in this range are $k = 3, 4, 5, 6, \ldots, 12, 13, 14, 15$, of which there are 13.

(b) Suppose that Karuna and Jorge meet for the first time after t_1 seconds and for the second time after t_2 seconds.
When they meet for the first time, Karuna has run partway from A to B and Jorge has run partway from B to A.



At this instant, the sum of the distances that they have run equals the total distance from A to B.
Since Karuna runs at 6 m/s for these t_1 seconds, she has run $6t_1$ m.
Since Jorge runs at 5 m/s for these t_1 seconds, he has run $5t_1$ m.
Therefore, $6t_1 + 5t_1 = 297$ and so $11t_1 = 297$ or $t_1 = 27$.

When they meet for the second time, Karuna has run from A to B and is running back to A and Jorge has run from B to A and is running back to B. This is because Jorge gets to A halfway through his run before Karuna gets back to A at the end of her run.



Since they each finish running after 99 seconds, then each has $99 - t_2$ seconds left to run.
At this instant, the sum of the distances that they have left to run equals the total distance from A to B.
Since Karuna runs at 6 m/s for these $(99 - t_2)$ seconds, she has to run $6(99 - t_2)$ m.
Since Jorge runs at 7.5 m/s for these $(99 - t_2)$ seconds, he has to run $7.5(99 - t_2)$ m.
Therefore, $6(99 - t_2) + 7.5(99 - t_2) = 297$ and so $13.5(99 - t_2) = 297$ or $99 - t_2 = 22$ and so $t_2 = 77$.

Alternatively, to calculate the value of t_2, we note that when Karuna and Jorge meet for the second time, they have each run the distance from A to B one full time and are on their return trips.
This means that they have each run the full distance from A to B once and the distances that they have run on their return trip add up to another full distance from A to B, for a total distance of $3 \cdot 297$ m $= 891$ m.
Karuna has run at 6 m/s for t_2 seconds, for a total distance of $6t_2$ m.
Jorge ran the first 297 m at 5 m/s, which took $\frac{297}{5}$ s and ran the remaining $(t_2 - \frac{297}{5})$ seconds at 7.5 m/s, for a total distance of $\left(297 + 7.5(t_2 - \frac{297}{5})\right)$ m.
Therefore,

$$\begin{aligned} 6t_2 + 297 + 7.5(t_2 - \tfrac{297}{5}) &= 891 \\ 13.5t_2 &= 891 - 297 + 7.5 \cdot \tfrac{297}{5} \\ 13.5t_2 &= 1039.5 \\ t_2 &= 77 \end{aligned}$$

Therefore, Karuna and Jorge meet after 27 seconds and after 77 seconds.

7. (a) *Solution 1*

Among a group of n people, there are $\dfrac{n(n-1)}{2}$ ways of choosing a pair of these people:

There are n people that can be chosen first.
For each of these n people, there are $n-1$ people that can be chosen second.
This gives $n(n-1)$ orderings of two people.
Each pair is counted twice (given two people A and B, we have counted both the pair AB and the pair BA), so the total number of pairs is $\dfrac{n(n-1)}{2}$.

We label the four canoes W, X, Y, and Z.
First, we determine the total number of ways to put the 8 people in the 4 canoes.
We choose 2 people to put in W. There are $\dfrac{8\cdot 7}{2}$ pairs. This leaves 6 people for the remaining 3 canoes.
Next, we choose 2 people to put in X. There are $\dfrac{6\cdot 5}{2}$ pairs. This leaves 4 people for the remaining 2 canoes.
Next, we choose 2 people to put in Y. There are $\dfrac{4\cdot 3}{2}$ pairs. This leaves 2 people for the remaining canoe.
There is now 1 way to put the remaining people in Z.
Therefore, there are

$$\frac{8\cdot 7}{2}\cdot\frac{6\cdot 5}{2}\cdot\frac{4\cdot 3}{2}=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{2^3}=7\cdot 6\cdot 5\cdot 4\cdot 3$$

ways to put the 8 people in the 4 canoes.

Now, we determine the number of ways in which no two of Barry, Carrie and Mary will be in the same canoe.
There are 4 possible canoes in which Barry can go.
There are then 3 possible canoes in which Carrie can go, because she cannot go in the same canoe as Barry.
There are then 2 possible canoes in which Mary can go, because she cannot go in the same canoe as Barry or Carrie.
This leaves 5 people left to put in the canoes.
There are 5 choices of the person that can go with Barry, and then 4 choices of the person that can go with Carrie, and then 3 choices of the person that can go with Mary.
The remaining 2 people are put in the remaining empty canoe.
This means that there are $4\cdot 3\cdot 2\cdot 5\cdot 4\cdot 3$ ways in which the 8 people can be put in 4 canoes so that no two of Barry, Carrie and Mary are in the same canoe.

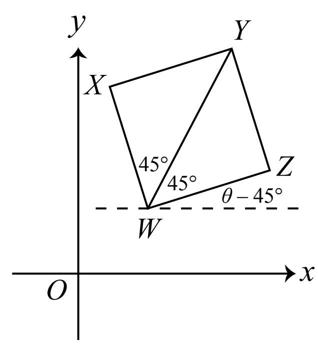
Therefore, the probability that no two of Barry, Carrie and Mary are in the same canoe is $\dfrac{4\cdot 3\cdot 2\cdot 5\cdot 4\cdot 3}{7\cdot 6\cdot 5\cdot 4\cdot 3}=\dfrac{4\cdot 3\cdot 2}{7\cdot 6}=\dfrac{24}{42}=\dfrac{4}{7}$.

Solution 2

Let p be the probability that two of Barry, Carrie and Mary are in the same canoe.
The answer to the original problem will be $1 - p$.
Let q be the probability that Barry and Carrie are in the same canoe.
By symmetry, the probability that Barry and Mary are in the same canoe also equals q as does the probability that Carrie and Mary are in the same canoe.
This means that $p = 3q$.
So we calculate q.
To do this, we put Barry in a canoe. Since there are 7 possible people who can go in the canoe with him, then the probability that Carrie is in the canoe with him equals $\frac{1}{7}$. The other 6 people can be put in the canoes in any way.
This means that the probability that Barry and Carrie are in the same canoe is $q = \frac{1}{7}$.
Therefore, the probability that no two of Barry, Carrie and Mary are in the same canoe is $1 - 3 \cdot \frac{1}{7}$ or $\frac{4}{7}$.

(b) *Solution 1*

Suppose that WY makes an angle of θ with the horizontal.



Since the slope of WY is 2, then $\tan\theta = 2$, since the tangent of an angle equals the slope of a line that makes this angle with the horizontal.
Since $\tan\theta = 2 > 1 = \tan 45^\circ$, then $\theta > 45^\circ$.
Now WY bisects $\angle ZWX$, which is a right-angle.
Therefore, $\angle ZWY = \angle YWX = 45^\circ$.
Therefore, WX makes an angle of $\theta + 45^\circ$ with the horizontal and WZ makes an angle of $\theta - 45^\circ$ with the horizontal. Since $\theta > 45^\circ$, then $\theta - 45^\circ > 0$ and $\theta + 45^\circ > 90^\circ$.
We note that since WZ and XY are parallel, then the slope of XY equals the slope of WZ.
To calculate the slopes of WX and WZ, we can calculate $\tan(\theta + 45^\circ)$ and $\tan(\theta - 45^\circ)$.
Using the facts that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, we obtain:

$$\tan(\theta + 45^\circ) = \frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta\tan 45^\circ} = \frac{2+1}{1-(2)(1)} = -3$$
$$\tan(\theta - 45^\circ) = \frac{\tan\theta - \tan 45^\circ}{1 - \tan\theta\tan 45^\circ} = \frac{2-1}{1+(2)(1)} = \frac{1}{3}$$

Therefore, the sum of the slopes of WX and XY is $-3 + \dfrac{1}{3} = -\dfrac{8}{3}$.

Solution 2

Consider a square $WXYZ$ whose diagonal WY has slope 2.

Translate this square so that W is at the origin $(0,0)$. Translating a shape in the plane does not affect the slopes of any line segments.

Let the coordinates of Y be $(2a, 2b)$ for some non-zero numbers a and b.

Since the slope of WY is 2, then $\dfrac{2b-0}{2a-0} = 2$ and so $2b = 4a$ or $b = 2a$.

Thus, the coordinates of Y can be written as $(2a, 4a)$.

Let C be the centre of square $WXYZ$.

Then C is the midpoint of WY, so C has coordinates $(a, 2a)$.

We find the slopes of WX and XY by finding the coordinates of X.

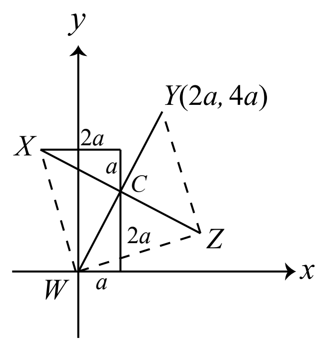
Consider the segment XC.

Since the diagonals of a square are perpendicular, then XC is perpendicular to WC.

Since the slope of WC is 2, then the slopes of XC and ZC are $-\frac{1}{2}$.

Since the diagonals of a square are equal in length and C is the midpoint of both diagonals, then $XC = WC$.

Since WC and XC are perpendicular and equal in length, then the "rise/run triangle" above XC will be a 90° rotation of the "rise/run triangle" below WC.



This is because these triangles are congruent (each is right-angled, their hypotenuses are of equal length, and their remaining angles are equal) and their hypotenuses are perpendicular.

In this diagram, we have assumed that X is to the left of W and Z is to the right of W. Since the slopes of parallel sides are equal, it does not matter which vertex is labelled X and which is labelled Z. We would obtain the same two slopes, but in a different order.

To get from $W(0,0)$ to $C(a, 2a)$, we go up $2a$ and right a.

Thus, to get from $C(a, 2a)$ to X, we go left $2a$ and up a.

Therefore, the coordinates of X are $(a - 2a, 2a + a)$ or $(-a, 3a)$.

Thus, the slope of WX is $\dfrac{3a-0}{-a-0} = -3$.

Since XY is perpendicular to WX, then its slope is the negative reciprocal of -3, which is $\dfrac{1}{3}$.

The sum of the slopes of WX and XY is $-3 + \dfrac{1}{3} = -\dfrac{8}{3}$.

8. (a) Since the base of a logarithm must be positive and cannot equal 1, then $x > 0$ and $x \neq \frac{1}{2}$ and $x \neq \frac{1}{3}$.
This tells us that $\log 2x$ and $\log 3x$ exist and do not equal 0, which we will need shortly when we apply the change of base formula.
We note further that $48 = 2^4 \cdot 3$ and $162 = 3^4 \cdot 2$ and $\sqrt[3]{3} = 3^{1/3}$ and $\sqrt[3]{2} = 2^{1/3}$.
Using logarithm rules, the following equations are equivalent:

$$\begin{aligned}
\log_{2x}(48\sqrt[3]{3}) &= \log_{3x}(162\sqrt[3]{2}) \\
\frac{\log(2^4 \cdot 3 \cdot 3^{1/3})}{\log 2x} &= \frac{\log(3^4 \cdot 2 \cdot 2^{1/3})}{\log 3x} \qquad \text{(change of base formula)} \\
\frac{\log(2^4 \cdot 3^{4/3})}{\log 2 + \log x} &= \frac{\log(3^4 \cdot 2^{4/3})}{\log 3 + \log x} \qquad (\log ab = \log a + \log b) \\
\frac{\log(2^4) + \log(3^{4/3})}{\log 2 + \log x} &= \frac{\log(3^4) + \log(2^{4/3})}{\log 3 + \log x} \qquad (\log ab = \log a + \log b) \\
\frac{4\log 2 + \frac{4}{3}\log 3}{\log 2 + \log x} &= \frac{4\log 3 + \frac{4}{3}\log 2}{\log 3 + \log x} \qquad (\log(a^c) = c\log a)
\end{aligned}$$

Cross-multiplying, we obtain

$$(4\log 2 + \tfrac{4}{3}\log 3)(\log 3 + \log x) = (4\log 3 + \tfrac{4}{3}\log 2)(\log 2 + \log x)$$

Expanding the left side, we obtain

$$4\log 2\log 3 + \tfrac{4}{3}(\log 3)^2 + (4\log 2 + \tfrac{4}{3}\log 3)\log x$$

Expanding the right side, we obtain

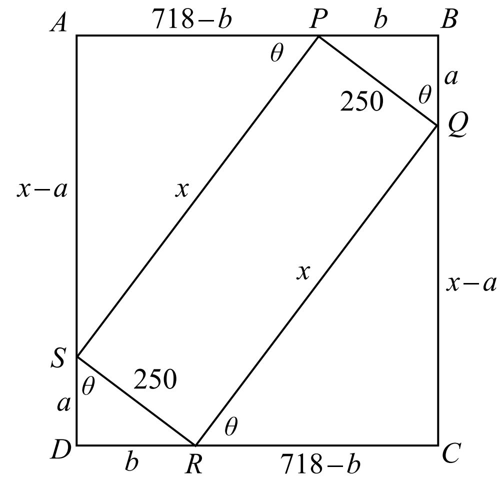
$$4\log 3\log 2 + \tfrac{4}{3}(\log 2)^2 + (4\log 3 + \tfrac{4}{3}\log 2)\log x$$

Simplifying and factoring, we obtain the following equivalent equations:

$$\begin{aligned}
\tfrac{4}{3}(\log 3)^2 - \tfrac{4}{3}(\log 2)^2 &= \log x(4\log 3 + \tfrac{4}{3}\log 2 - 4\log 2 - \tfrac{4}{3}\log 3) \\
\tfrac{4}{3}(\log 3)^2 - \tfrac{4}{3}(\log 2)^2 &= \log x\left(\tfrac{8}{3}\log 3 - \tfrac{8}{3}\log 2\right) \\
(\log 3)^2 - (\log 2)^2 &= 2\log x(\log 3 - \log 2) \\
\log x &= \frac{(\log 3)^2 - (\log 2)^2}{2(\log 3 - \log 2)} \\
\log x &= \frac{(\log 3 - \log 2)(\log 3 + \log 2)}{2(\log 3 - \log 2)} \\
\log x &= \frac{\log 3 + \log 2}{2} \\
\log x &= \tfrac{1}{2}\log 6 \\
\log x &= \log(\sqrt{6})
\end{aligned}$$

and so $x = \sqrt{6}$.

(b) Let $BC = x$, $PB = b$, and $BQ = a$.
Since $BC = x$, then $AD = PS = QR = x$.
Since $BC = x$ and $BQ = a$, then $QC = x - a$.
Since $AB = 718$ and $PB = b$, then $AP = 718 - b$.
Note that $PQ = SR = 250$.
Let $\angle BQP = \theta$.
Since $\triangle PBQ$ is right-angled at B, then $\angle BPQ = 90^\circ - \theta$.
Since BQC is a straight angle and $\angle PQR = 90^\circ$, then $\angle RQC = 180^\circ - 90^\circ - \theta = 90^\circ - \theta$.
Since APB is a straight angle and $\angle SPQ = 90^\circ$, then $\angle APS = 180^\circ - 90^\circ - (90^\circ - \theta) = \theta$.
Since $\triangle SAP$ and $\triangle QCR$ are each right-angled and have another angle in common with $\triangle PBQ$, then these three triangles are similar.



Continuing in the same way, we can show that $\triangle RDS$ is also similar to these three triangles.
Since $RS = PQ$, then $\triangle RDS$ is actually congruent to $\triangle PBQ$ (angle-side-angle).
Similarly, $\triangle SAP$ is congruent to $\triangle QCR$.
In particular, this means that $AS = x - a$, $SD = a$, $DR = b$, and $RC = 718 - b$.
Since $\triangle SAP$ and $\triangle PBQ$ are similar, then $\dfrac{SA}{PB} = \dfrac{AP}{BQ} = \dfrac{SP}{PQ}$.
Thus, $\dfrac{x-a}{b} = \dfrac{718-b}{a} = \dfrac{x}{250}$.
Also, by the Pythagorean Theorem in $\triangle PBQ$, we obtain $a^2 + b^2 = 250^2$.
By the Pythagorean Theorem in $\triangle SAP$,

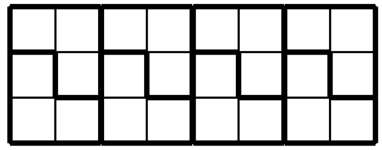
$$\begin{aligned} x^2 &= (x-a)^2 + (718-b)^2 \\ x^2 &= x^2 - 2ax + a^2 + (718-b)^2 \\ 0 &= -2ax + a^2 + (718-b)^2 \qquad (*) \end{aligned}$$

Since $a^2 + b^2 = 250^2$, then $a^2 = 250^2 - b^2$.
Since $\dfrac{718-b}{a} = \dfrac{x}{250}$, then $ax = 250(718 - b)$.
Therefore, substituting into $(*)$, we obtain

$$\begin{aligned} 0 &= -2(250)(718-b) + 250^2 - b^2 + (718-b)^2 \\ b^2 &= 250^2 - 2(250)(718-b) + (718-b)^2 \\ b^2 &= ((718-b) - 250)^2 \qquad (\text{since } y^2 - 2yz + z^2 = (y-z)^2) \\ b^2 &= (468-b)^2 \\ b &= 468 - b \qquad (\text{since } b \neq b - 468) \\ 2b &= 468 \\ b &= 234 \end{aligned}$$

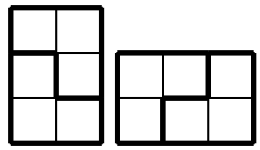
Therefore, $a^2 = 250^2 - b^2 = 250^2 - 234^2 = (250+234)(250-234) = 484 \cdot 16 = 22^2 \cdot 4^2 = 88^2$ and so $a = 88$.
Finally, $x = \dfrac{250(718-b)}{a} = \dfrac{250 \cdot 484}{88} = 1375$. Therefore, $BC = 1375$.

9. (a) Here is a tiling of a 3×8 rectangle:



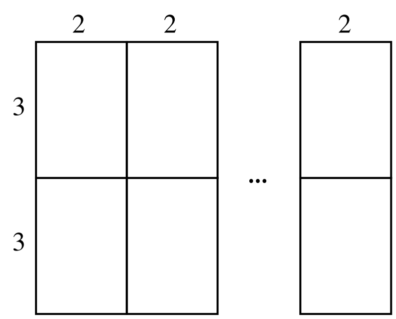
There are many other tilings.

(b) First, we note that it is possible to tile each of a 3×2 and a 2×3 rectangle:



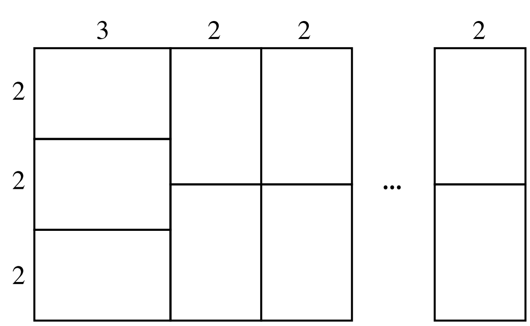
Next, we note that it is not possible to tile a 6×1 rectangle because each of the triominos needs a width of at least 2 to be placed.
Finally, we show that it is possible to tile a $6 \times W$ rectangle for every integer $W \geq 2$.
To do this, we show that such a $6 \times W$ rectangle can be made up from 3×2 and 2×3 rectangles. Since each of these types of rectangles can be tiled with triominos, then the larger rectangle can be tiled with triominos by combining these tilings.

Case 1: W is even
Suppose that $W = 2k$ for some positive integer k.
We build a $6 \times 2k$ rectangle by placing k 6×2 rectangles side by side.
Each 6×2 rectangle is built by stacking two 3×2 rectangles on top of each other.



Therefore, each such rectangle can be tiled.

Case 2: W is odd, $W \geq 3$
Suppose that $W = 2k + 1$ for some positive integer k.
We build a $6 \times (2k + 1)$ rectangle by building a 6×3 rectangle and then putting $k - 1$ 6×2 rectangles next to it. Note that $k - 1 \geq 0$ since $k \geq 1$ and that $2k + 1 = 3 + 2(k - 1)$.
The 6×3 rectangle is built by stacking three 2×3 rectangles on top of each other.
Each 6×2 rectangle is built by stacking two 3×2 rectangles on top of each other.



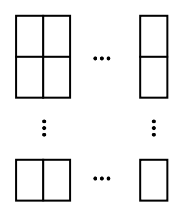
Therefore, each such rectangle can be tiled.

Thus, a $6 \times W$ rectangle can be tiled with triominos exactly when $W \geq 2$.

(c) Suppose that (H, W) is a pair of integers with $H \geq 4$ and $W \geq 4$.
Since the area of each triomino is 3, then the area of any rectangle that can be tiled must be a multiple of 3 since it is completely covered by triominos with area 3.
Since the area of an $H \times W$ rectangle is HW, then we need HW to be a multiple of 3, which means that at least one of H and W is a multiple of 3.
Since a rectangle that is $a \times b$ can be tiled if and only if a rectangle that is $b \times a$ can be tiled (we see this by rotating the tilings by 90° as we did with the 3×2 and 2×3 rectangles above), then we may assume without loss of generality that H is divisible by 3.
We show that if H is divisible by 3, then every $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled.

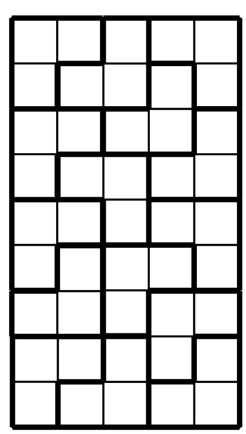
Case 1: H is divisible by 3, H is even
Here, H is a multiple of 6, say $H = 6m$ for some positive integer m.
Since $W \geq 4$, we know that a $6 \times W$ rectangle can be tiled.
By stacking m $6 \times W$ rectangles on top of each other, we obtain a $6m \times W$ rectangle.
Since each $6 \times W$ rectangle can be tiled, then the $6m \times W$ rectangle can be tiled.

Case 2: H is divisible by 3, H is odd, W is even
Suppose that $H = 3q$ for some odd positive integer q and $W = 2r$ for some positive integer r.
To tile a $3q \times 2r$ rectangle, we combine qr 3×2 rectangles in q rows and r columns:



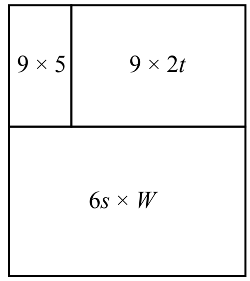
Therefore, every rectangle in this case can be tiled. (Note that in this case the fact that q was odd was not important.)

Case 3: H is divisible by 3, H is odd, W is odd
Since $H \geq 4$ and $W \geq 4$, the rectangle with the smallest values of H and W is 9×5 which can be tiled as shown:



(There are also other ways to tile this rectangle.)
Since H is an odd multiple of 3 and $H \geq 4$, we can write $H = 9 + 6s$ for some integer $s \geq 0$.

Since W is odd and $W \geq 5$, we can write $W = 5 + 2t$ for some integer $t \geq 0$.
Thus, the $H \times W$ rectangle is $(9 + 6s) \times (5 + 2t)$.
We break this rectangle into three rectangles – one that is 9×5, one that is $9 \times 2t$, and one that is $6s \times W$:



(If $s = 0$ or $t = 0$, there will be fewer than three rectangles.)
The 9×5 rectangle can be tiled as we showed earlier.
If $t > 0$, the $9 \times 2t$ rectangle can be tiled as seen in Case 2.
If $s > 0$, the $6s \times W$ rectangle can be tiled as seen in Case 1.
Therefore, the $H \times W$ rectangle can be tiled.

Through these three cases, we have shown that any $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled when H is a multiple of 3.
Since the roles of H and W can be interchanged and since at least one of H and W must be a multiple of 3, then an $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled exactly when at least one of H and W is a multiple of 3.

10. (a) We draw part of the array using the information that $A_0 = A_1 = A_2 = 0$ and $A_3 = 1$:

$\cdots$	A_0	A_1	A_2	A_3	A_4	A_5	$\cdots$
$\cdots$	B_0	B_1	B_2	B_3	B_4	B_5	$\cdots$

$\cdots$	0	0	0	1	A_4	A_5	$\cdots$
$\cdots$	B_0	B_1	B_2	B_3	B_4	B_5	$\cdots$

Since A_1 is the average of A_0, B_1 and A_2, then $A_1 = \dfrac{A_0 + B_1 + A_2}{3}$ or $3A_1 = A_0 + B_1 + A_2$.
Thus, $3(0) = 0 + B_1 + 0$ and so $B_1 = 0$.
Since A_2 is the average of A_1, B_2 and A_3, then $3A_2 = A_1 + B_2 + A_3$ and so $3(0) = 0 + B_2 + 1$ which gives $B_2 = -1$.
Since B_2 is the average of B_1, A_2 and B_3, then $3B_2 = B_1 + A_2 + B_3$ and so $3(-1) = 0 + 0 + B_3$ which gives $B_3 = -3$.
So far, this gives

$\cdots$	0	0	0	1	A_4	A_5	$\cdots$
$\cdots$	B_0	0	-1	-3	B_4	B_5	$\cdots$

Since A_3 is the average of A_2, B_3 and A_4, then $3A_3 = A_2 + B_3 + A_4$ and so $3(1) = 0 + (-3) + A_4$ which gives $A_4 = 6$.

(b) We draw part of the array:

$$\begin{array}{c|c|c|c|c} \cdots & A_{k-1} & A_k & A_{k+1} & \cdots \\ \hline \cdots & B_{k-1} & B_k & B_{k+1} & \cdots \end{array}$$

Then

$$\begin{aligned} 3S_k &= 3A_k + 3B_k \\ &= 3\left(\frac{A_{k-1} + B_k + A_{k+1}}{3}\right) + 3\left(\frac{B_{k-1} + A_k + B_{k+1}}{3}\right) \\ &= A_{k-1} + B_k + A_{k+1} + B_{k-1} + A_k + B_{k+1} \\ &= (A_{k-1} + B_{k-1}) + (A_k + B_k) + (A_{k+1} + B_{k+1}) \\ &= S_{k-1} + S_k + S_{k+1} \end{aligned}$$

Since $3S_k = S_{k-1} + S_k + S_{k+1}$, then $S_{k+1} = 2S_k - S_{k-1}$.

(c) *Proof of statement (P)*
Suppose that all of the entries in the array are positive integers.
Assume that not all of the entries in the array are equal.
Since all of the entries are positive integers, there must be a minimum entry. Let m be the minimum of all of the entries in the array.
Choose an entry in the array equal to m, say $A_r = m$ for some integer r. The same argument can be applied with $B_r = m$ if there are no entries equal to m in the top row.
If not all of the entries A_j are equal to m, then by moving one direction or the other along the row we will get to some point where $A_t = m$ for some integer t but one of its neighbours is not equal to m. (If this were not to happen, then all of the entries in both directions would be equal to m.)
If all of the entries A_j are equal to m, then since not all of the entries in the array are equal to m, then there will be an entry B_t which is not equal to m.
In other words, since not all of the entries in the array are equal, then there exists an integer t for which $A_t = m$ and not all of A_{t-1}, A_{t+1}, B_t are equal to m.
But $3m = 3A_t$ and $3A_t = A_{t-1} + B_t + A_{t+1}$ so $3m = A_{t-1} + B_t + A_{t+1}$.
Since not all of A_{t-1}, B_t and A_{t+1} are equal to m and each is at least m, then one of these entries will be greater than m.
This means that $A_{t-1} + B_t + A_{t+1} \geq m + m + (m+1) = 3m + 1 > 3m$, which is a contradiction.
Therefore our assumption that not all of the entries are equal must be false, which means that all of the entries are equal, which proves statement (P).

Proof of statement (Q)
Suppose that all of the entries are positive real numbers.
Assume that not all of the entries in the array are equal.
As in (b), define $S_k = A_k + B_k$ for each integer k.
Also, define $D_k = A_k - B_k$ for each integer k.

Step 1: Prove that the numbers S_k form an arithmetic sequence
From (b), $S_{k+1} = 2S_k - S_{k-1}$.
Re-arranging, we see $S_{k+1} - S_k = S_k - S_{k-1}$ for each integer k, which means that the differences between consecutive pairs of terms are equal.
Since this is true for all integers k, then the difference between each pair of consecutive

terms through the whole sequence is constant, which means that the sequence is an arithmetic sequence.

Step 2: Prove that S_k is constant

Suppose that $S_0 = c$. Since $A_0 > 0$ and $B_0 > 0$, then $S_0 = c > 0$.
Since the terms S_k form an arithmetic sequence, then the sequence is either constant, increasing or decreasing.
If the sequence of terms S_k is increasing, then the common difference $d = S_1 - S_0$ is positive.
Note that $S_{-1} = c - d$, $S_{-2} = c - 2d$, and so on.
Since c and d are constant, then if we move far enough back along the sequence, eventually S_t will be negative for some integer t. This is a contradiction since $A_t > 0$ and $B_t > 0$ and $S_t = A_t + B_t$.
Thus, the sequence cannot be increasing.
If the sequence of terms S_k is decreasing, then the common difference $d = S_1 - S_0$ is negative.
Note that $S_1 = c + d$, $S_2 = c + 2d$, and so on.
Since c and d are constant, then if we move far along the sequence, eventually S_t will be negative for some integer t. This is also a contradiction since $A_t > 0$ and $B_t > 0$ and $S_t = A_t + B_t$.
Thus, the sequence cannot be decreasing.
Therefore, since all of the entries are positive and the sequence S_k is arithmetic, then S_k is constant, say $S_k = c > 0$ for all integers k.

Step 3: Determine range of possible values for D_k

We note that $S_k = A_k + B_k = c$ for all integers k and $A_k > 0$ and $B_k > 0$.
Since $A_k > 0$, then $B_k = S_k - A_k = c - A_k < c$.
Similarly, $A_k < c$.
Therefore, $0 < A_k < c$ and $0 < B_k < c$.
Since $D_k = A_k - B_k$, then $D_k < c - 0 = c$ and $D_k > 0 - c = -c$.
In other words, $-c < D_k < c$.

Step 4: $D_{k+1} = 4D_k - D_{k-1}$

Using a similar approach to our solution to (b),

$$\begin{aligned}
3D_k &= 3A_k - 3B_k \\
3D_k &= (A_{k-1} + B_k + A_{k+1}) - (B_{k-1} + A_k + B_{k+1}) \\
3D_k &= (A_{k+1} - B_{k+1}) + (A_{k-1} - B_{k-1}) - (A_k - B_k) \\
3D_k &= D_{k+1} + D_{k-1} - D_k \\
4D_k - D_{k-1} &= D_{k+1}
\end{aligned}$$

as required.

Step 5: Final contradiction

We want to show that $D_k = 0$ for all integers k.
This will show that $A_k = B_k$ for all integers k.
Since $S_k = A_k + B_k = c$ for all integers k, then this would show that $A_k = B_k = \frac{1}{2}c$ for all integers k, meaning that all entries in the array are equal.
Suppose that $D_k \neq 0$ for some integer k.
We may assume that $D_0 \neq 0$. (If $D_0 = 0$, then because the array is infinite in both directions, we can shift the numbering of the array so that a column where $D_k \neq 0$ is

labelled column 0.)
Thus, $D_0 > 0$ or $D_0 < 0$.
We may assume that $D_0 > 0$. (If $D_0 < 0$, then we can switch the bottom and top rows of the array so that D_0 becomes positive.)
Suppose that $D_1 \geq D_0 > 0$.
Then $D_2 = 4D_1 - D_0 \geq 4D_1 - D_1 = 3D_1$. Since $D_1 > 0$, this also means that $D_2 > D_1 > 0$.
Similarly, $D_3 = 4D_2 - D_1 \geq 4D_2 - D_2 = 3D_2 > D_2 > 0$. Since $D_2 \geq 3D_1$, then $D_3 \geq 9D_1$.
Continuing in this way, we see that $D_4 \geq 27D_1$ and $D_5 \geq 81D_1$ and so on, with $D_k \geq 3^{k-1}D_1$ for each positive integer $k \geq 2$. Since the value of D_1 is a fixed positive real number and $D_k < c$ for all integers k, this is a contradiction, because the sequence of values 3^{k-1} grows without bound.
The other possibility is that $D_1 < D_0$.
Here, we re-arrange $D_{k+1} = 4D_k - D_{k-1}$ to obtain $D_{k-1} = 4D_k - D_{k+1}$.
Thus, $D_{-1} = 4D_0 - D_1 > 4D_0 - D_0 = 3D_0 > D_0 > 0$.
Extending this using a similar method, we see that $D_{-j} > 3^j D_0$ for all positive integers j which will lead to the same contradiction as above.
Therefore, a contradiction is obtained in all cases and so it cannot be the case that $D_k \neq 0$ for some integer k.

Since $D_k = 0$ and $S_k = c$ for all integers k, then $A_k = B_k = \frac{1}{2}c$ for all integers k, which means that all entries in the array are equal.