The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 11, 2018
(in North America and South America)

Thursday, April 12, 2018
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

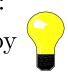
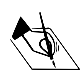
Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by 
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked (lightbulb icon), place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked (pencil icon), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling
Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

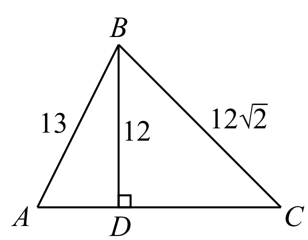
1. (a) If $x = 11$, what is the value of $x + (x + 1) + (x + 2) + (x + 3)$?

   (b) If $\frac{a}{6} + \frac{6}{18} = 1$, what is the value of a?

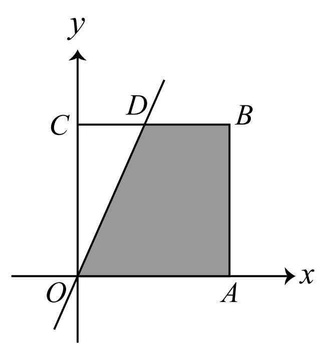
   (c) The total cost of one chocolate bar and two identical packs of gum is \$4.15. One chocolate bar costs \$1.00 more than one pack of gum. Determine the cost of one chocolate bar.

2. (a) A five-digit integer is made using each of the digits $1, 3, 5, 7, 9$. The integer is greater than 80 000 and less than 92 000. The units (ones) digit is 3. The hundreds and tens digits, in that order, form a two-digit integer that is divisible by 5. What is the five-digit integer?

   (b) In the diagram, point D is on AC so that BD is perpendicular to AC. Also, $AB = 13$, $BC = 12\sqrt{2}$ and $BD = 12$. What is the length of AC?



   (c) In the diagram, square $OABC$ has side length 6. The line with equation $y = 2x$ intersects CB at D. Determine the area of the shaded region.



3\. (a) What is the value of $\left(\sqrt{4+\sqrt{4}}\right)^4$?

(b) There is exactly one pair (x, y) of positive integers for which $\sqrt{23-x} = 8 - y^2$. What is this pair (x, y)?

(c) The line with equation $y = mx + 2$ intersects the parabola with equation $y = ax^2 + 5x - 2$ at the points $P(1, 5)$ and Q. Determine

(i) the value of m,

(ii) the value of a, and

(iii) the coordinates of Q.

4\. (a) The positive integers 34 and 80 have exactly two positive common divisors, namely 1 and 2. How many positive integers n with $1 \leq n \leq 30$ have the property that n and 80 have exactly two positive common divisors?

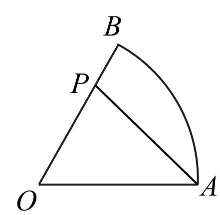
(b) A function f is defined so that

- $f(1) = 1$,
- if n is an even positive integer, then $f(n) = f\left(\frac{1}{2}n\right)$, and
- if n is an odd positive integer with $n > 1$, then $f(n) = f(n-1) + 1$.

For example, $f(34) = f(17)$ and $f(17) = f(16) + 1$.
Determine the value of $f(50)$.

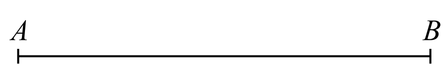
5\. (a) The perimeter of equilateral $\triangle PQR$ is 12. The perimeter of regular hexagon $STUVWX$ is also 12. What is the ratio of the area of $\triangle PQR$ to the area of $STUVWX$?

(b) In the diagram, sector AOB is $\frac{1}{6}$ of an entire circle with radius $AO = BO = 18$. The sector is cut into two regions with a single straight cut through A and point P on OB. The areas of the two regions are equal. Determine the length of OP.



6\. (a) For how many integers k with $0 < k < 18$ is $\dfrac{5\sin(10k^\circ) - 2}{\sin^2(10k^\circ)} \geq 2$?

(b) In the diagram, a straight, flat road joins A to B.



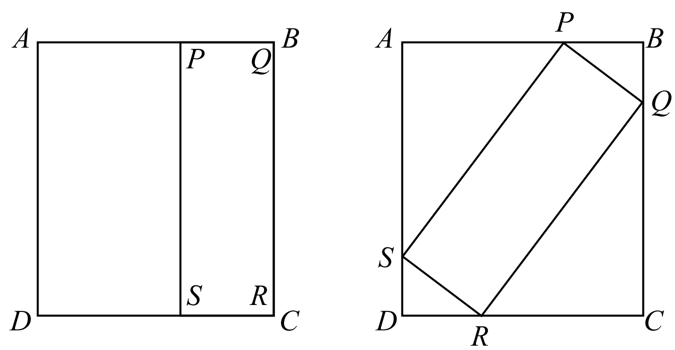
Karuna runs from A to B, turns around instantly, and runs back to A. Karuna runs at 6 m/s. Starting at the same time as Karuna, Jorge runs from B to A, turns around instantly, and runs back to B. Jorge runs from B to A at 5 m/s and from A to B at 7.5 m/s. The distance from A to B is 297 m and each runner takes exactly 99 s to run their route. Determine the two values of t for which Karuna and Jorge are at the same place on the road after running for t seconds.

7. (a) Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probability that no two of Barry, Carrie and Mary will be in the same canoe?

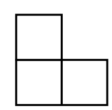
(b) Diagonal WY of square $WXYZ$ has slope 2. Determine the sum of the slopes of WX and XY.

8. (a) Determine all values of x such that $\log_{2x}(48\sqrt[3]{3}) = \log_{3x}(162\sqrt[3]{2})$.

(b) In the diagram, rectangle $PQRS$ is placed inside rectangle $ABCD$ in two different ways: first, with Q at B and R at C; second, with P on AB, Q on BC, R on CD, and S on DA.



If $AB = 718$ and $PQ = 250$, determine the length of BC.

9. An L-shaped triomino is composed of three unit squares, as shown: 

Suppose that H and W are positive integers. An $H \times W$ rectangle can be *tiled* if the rectangle can be completely covered with non-overlapping copies of this triomino (each of which can be rotated and/or translated) and the sum of the areas of these non-overlapping triominos equals the area of the rectangle (that is, no triomino is partly outside the rectangle). If such a rectangle can be tiled, a *tiling* is a specific configuration of triominos that tile the rectangle.

(a) Draw a tiling of a 3×8 rectangle.

(b) Determine, with justification, all integers W for which a $6 \times W$ rectangle can be tiled.

(c) Determine, with justification, all pairs (H, W) of integers with $H \geq 4$ and $W \geq 4$ for which an $H \times W$ rectangle can be tiled.

10. In an infinite array with two rows, the numbers in the top row are denoted $\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots$ and the numbers in the bottom row are denoted $\ldots, B_{-2}, B_{-1}, B_0, B_1, B_2, \ldots$. For each integer k, the entry A_k is directly above the entry B_k in the array, as shown:

$\ldots$	A_{-2}	A_{-1}	A_0	A_1	A_2	$\ldots$
$\ldots$	B_{-2}	B_{-1}	B_0	B_1	B_2	$\ldots$

For each integer k, A_k is the average of the entry to its left, the entry to its right, and the entry below it; similarly, each entry B_k is the average of the entry to its left, the entry to its right, and the entry above it.

(a) In one such array, $A_0 = A_1 = A_2 = 0$ and $A_3 = 1$.
Determine the value of A_4.
The maximum mark on this part is 2 marks.

(b) In another such array, we define $S_k = A_k + B_k$ for each integer k.
Prove that $S_{k+1} = 2S_k - S_{k-1}$ for each integer k.
The maximum mark on this part is 2 marks.

(c) Consider the following two statements about a third such array:

(P) If each entry is a positive integer, then all of the entries in the array are equal.

(Q) If each entry is a positive real number, then all of the entries in the array are equal.

Prove statement (Q).
The maximum mark on this part is 6 marks.

A complete proof of statement (Q) will earn the maximum of 6 marks for part (c), regardless of whether any attempt to prove (P) is made.
A complete proof of statement (P) will earn 2 of the 6 possible marks for part (c). In such a case, any further progress towards proving (Q) would be assessed for partial marks towards the remaining 4 marks.
Students who do not fully prove either (P) or (Q) will have their work assessed for partial marks.

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For students...

Thank you for writing the 2018 Euclid Contest! Each year, more than 240 000 students from more than 75 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2018 Canadian Senior Mathematics Contest, which will be written in November 2018.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2018/2019 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results