The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

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Canadian Senior Mathematics Contest

Wednesday, November 21, 2018
(in North America and South America)

Thursday, November 22, 2018
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

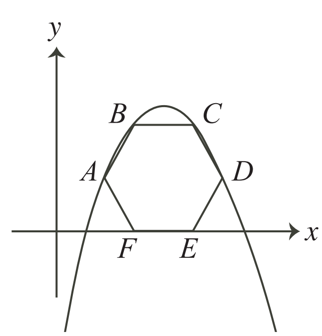
1. Paul has 6 boxes, each of which contains 12 trays. Paul also has 4 extra trays. If each tray can hold 8 apples, what is the largest possible number of apples that can be held by the 6 boxes and 4 extra trays?

2. A rabbit, a skunk and a turtle are running a race.
The skunk finishes the race in 6 minutes.
The rabbit runs 3 times as quickly as the skunk.
The rabbit runs 5 times as quickly as the turtle.
How long does the turtle take to finish the race?

3. A jar contains 6 crayons, of which 3 are red, 2 are blue, and 1 is green. Jakob reaches into the jar and randomly removes 2 of the crayons. What is the probability that both of these crayons are red?

4. Suppose that n is positive integer and that a is the integer equal to $\dfrac{10^{2n} - 1}{3(10^n + 1)}$.
If the sum of the digits of a is 567, what is the value of n?

5. In the diagram, $ABCDEF$ is a regular hexagon with side length 2. Points E and F are on the x-axis and points A, B, C, and D lie on a parabola. What is the distance between the two x-intercepts of the parabola?



6. Suppose that $0^\circ < A < 90^\circ$ and $0^\circ < B < 90^\circ$ and

$$(4+\tan^2 A)(5+\tan^2 B) = \sqrt{320}\tan A \tan B$$

Determine all possible values of $\cos A \sin B$.

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. Alexandra draws a letter A which stands on the x-axis.

   (a) The left side of the letter A lies along the line with equation $y = 3x + 6$. What is the x-intercept of the line with equation $y = 3x + 6$?

   (b) The right side of the letter A lies along the line L_2 and the letter is symmetric about the y-axis. What is the equation of line L_2?

   (c) Determine the area of the triangle formed by the x-axis and the left and right sides of the letter A.

   (d) Alexandra completes the letter A by adding to Figure 1. She draws the horizontal part of the letter A along the line $y = c$, as in Figure 2. The area of the shaded region inside the letter A and above the line with equation $y = c$ is $\frac{4}{9}$ of the total area of the region above the x-axis and between the left and right sides. Determine the value of c.

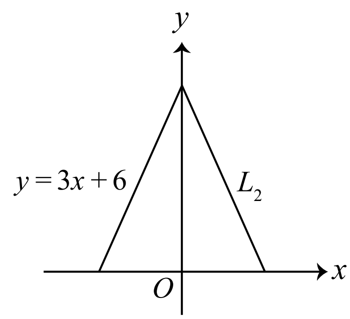


Figure 1

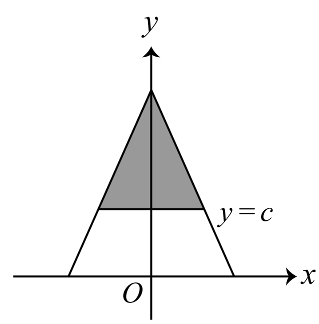


Figure 2

2. (a) Determine the positive integer x for which $\dfrac{1}{4} - \dfrac{1}{x} = \dfrac{1}{6}$.

   (b) Determine all pairs of positive integers (a, b) for which $ab - b + a - 1 = 4$.

   (c) Determine the number of pairs of positive integers (y, z) for which $\dfrac{1}{y} - \dfrac{1}{z} = \dfrac{1}{12}$.

   (d) Prove that, for every prime number p, there are at least two pairs (r, s) of positive integers for which $\dfrac{1}{r} - \dfrac{1}{s} = \dfrac{1}{p^2}$.

3. A *string of length* n is a sequence of n characters from a specified set. For example, $BCAAB$ is a string of length 5 with characters from the set $\{A, B, C\}$. A *substring* of a given string is a string of characters that occur consecutively and in order in the given string. For example, the string CA is a substring of $BCAAB$ but BA is not a substring of $BCAAB$.

   (a) List all strings of length 4 with characters from the set $\{A, B, C\}$ in which both the strings AB and BA occur as substrings. (For example, the string $ABAC$ should appear in your list.)

   (b) Determine the number of strings of length 7 with characters from the set $\{A, B, C\}$ in which CC occurs as a substring.

   (c) Let $f(n)$ be the number of strings of length n with characters from the set $\{A, B, C\}$ such that

   - CC occurs as a substring, and
   - if either AB or BA occurs as a substring then there is an occurrence of the substring CC to its left.

   (For example, when $n = 6$, the strings $CCAABC$ and $ACCBBB$ and $CCABCC$ satisfy the requirements, but the strings $BACCAB$ and $ACBBAB$ and $ACBCAC$ do not.) Prove that $f(2097)$ is a multiple of 97.