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2018 Canadian Intermediate Mathematics Contest

Wednesday, November 21, 2018
(in North America and South America)

Thursday, November 22, 2018
(outside of North America and South America)

Solutions

Part A

1. Since the angles in any triangle add to 180°, then

$$\begin{aligned} 60^\circ + (5x)^\circ + (3x)^\circ &= 180^\circ \\ 60 + 5x + 3x &= 180 \\ 8x &= 120 \end{aligned}$$

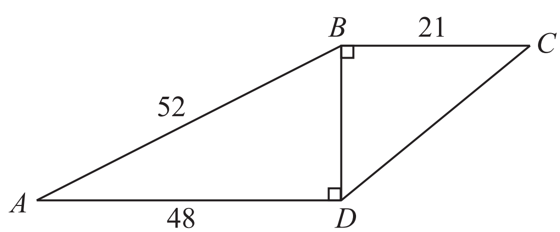
and so $x = 15$.

Answer: $x = 15$

2. Since 10% of the 500 animals are chickens, then $\frac{1}{10} \times 500 = 50$ of the animals are chickens.
The remaining $500 - 50 = 450$ animals are goats and cows.
Since there are twice as many goats as cows, then the total number of goats and cows is three times the number of cows.
Therefore, there are $\frac{1}{3} \times 450 = 150$ cows.
(Since there are 150 cows, then there are $2 \times 150 = 300$ goats. Combining with the 50 chickens, we have $150 + 300 + 50 = 500$ animals, as expected.)

Answer: 150 cows

3. Since $\triangle ADB$ and $\triangle CBD$ are right-angled, we can apply the Pythagorean Theorem in each triangle.



Therefore, $AB^2 = AD^2 + BD^2$ or $52^2 = 48^2 + BD^2$ and so $BD^2 = 52^2 - 48^2 = 2704 - 2304 = 400$.
Since $BD > 0$, then $BD = \sqrt{400} = 20$.
Furthermore, $DC^2 = BD^2 + BC^2 = 20^2 + 21^2 = 400 + 441 = 841$.
Since $DC > 0$, then $DC = \sqrt{841} = 29$.

Answer: $DC = 29$

4. Let x be the number in the bottom right corner of the 8×8 square.
Since there are 8 numbers across the bottom row of this square, then the number in the bottom left corner is $x - 7$. (The 8 numbers in the bottom row are $x - 7$, $x - 6$, $x - 5$, $x - 4$, $x - 3$, $x - 2$, $x - 1$, and x.)
In the larger grid, each number is 24 greater than the number directly above it.
The number in the bottom right corner of the 8×8 square is seven rows beneath the number in the upper right corner.
Since there are 24 numbers in each row, each number in the grid above the bottom row is 24 less than the number directly below it.
This means the number in the top right corner is $7 \times 24 = 168$ less than the number in the

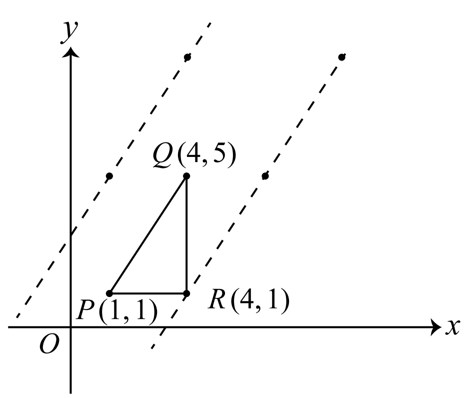
bottom right corner, as it is 7 rows above it.
Since the number in the bottom right corner is x, then the number in the top right corner is $x - 168$.
Moving to the left across the first row of the 8×8 square, the number in the top left corner is $(x - 168) - 7$ or $x - 175$.
Since the sum of the numbers in the four corners of the square is to be 1646, then

$$\begin{aligned} x + (x-7) + (x-168) + (x-175) &= 1646 \\ 4x - 350 &= 1646 \\ 4x &= 1996 \\ x &= 499 \end{aligned}$$

Therefore, the number in the bottom right corner is 499.
We should also verify that, starting with 499 in the bottom right corner, we can construct an 8×8 square that doesn't extend past the top or left border of the grid.
Since the bottom right number in the grid is 576, then the numbers up the right side of the grid are 576, 552, 528, 504, which means that, moving from right to left, 499 is sixth number of the 4th row from the bottom.
Since the grid is 24×24, then the square does not meet the left or top borders.

ANSWER: 499

5. Consider the point $R(4,1)$.
Here, PR is horizontal and RQ is vertical and so $\triangle PQR$ is right-angled at R.
Because $PR = 4 - 1 = 3$ and $RQ = 5 - 1 = 4$, then the area of $\triangle PQR$ is $\frac{1}{2} \times PR \times RQ$ which equals $\frac{1}{2} \times 3 \times 4$ or 6.
Therefore, the point $R(4,1)$ has the property that $\triangle PQR$ has area 6.
If a point X lies on the line through R parallel to PQ, then the perpendicular distance from X to the line through PQ is the same as the perpendicular distance from R to PQ. This means that the height of $\triangle PQX$ equals the height of $\triangle PQR$ and so the areas of these triangles are equal.



In other words, any point X on the line through R parallel to PQ has the property that the area of $\triangle PQX$ is 6.
To get from P to Q, we move 3 units to the right and 4 units up.
Therefore, starting at $R(4,1)$ and moving to the points $(4+3, 1+4) = (7,5)$ will get to another point on the line through $R(4,1)$ parallel to PQ (since moving right 3 and up 4 preserves the

slope of $\frac{4}{3}$.).
Similarly, the point $(7+3, 5+4) = (10, 9)$ is on this line.
In other words, the triangle with vertices at $(1, 1)$, $(4, 5)$ and $(7, 5)$ and the triangle with vertices at $(1, 1)$, $(4, 5)$ and $(10, 9)$ both have areas of 6.
(To do this more formally, we could determine that the equation of the line through R parallel to PQ is $y = \frac{4}{3}x - \frac{13}{3}$ and then check the possible integer x-coordinates of points on this line (between 0 and 10, inclusive) to get the points $(1, -3)$, $(4, 1)$, $(7, 5)$, $(10, 9)$, omitting $(1, -3)$ since -3 is not in the correct range).

Next, consider $S(1, 5)$.
Again, the area of $\triangle PQS$ is 6 because PS is vertical (with length 4) and QS is horizontal (with length 3).
Moving right 3 and up 4 from $(1, 5)$ gives the point $(4, 9)$.

Since we are told that there are five such points, then we have found all of the points and they are $(4, 1)$, $(7, 5)$, $(10, 9)$, $(1, 5)$, and $(4, 9)$.
(We could note that any point T for which the area of $\triangle PQT$ is 6 must lie on one of these two lines. This is because T must have a particular perpendicular distance to the line through PQ and must be on one side of this line or the other. Because these two lines are fixed, there are indeed only five points with these properties.)

Answer: $(4, 1)$, $(7, 5)$, $(10, 9)$, $(1, 5)$, and $(4, 9)$

6. We know that $1 \leq n \leq 20$ and $1 \leq k \leq 20$.
We focus on the possible values of k.
Consider $k = 20$.
Since there are 20 chairs, then moving 20 chairs around the circle moves each person back to their original seat.
Therefore, any configuration of people sitting in chairs is preserved by this movement.
In other words, n can take any value from 1 to 20, inclusive, when $k = 20$.
Thus, there are 20 pairs (n, k) that work when $k = 20$.

Consider $k = 10$.
Here, any people in chairs 1 to 10 move to chairs 11 to 20 and any people in chairs 11 to 20 move to chairs 1 to 10.
This means that the two halves (1 to 10 and 11 to 20) must contain the same number of people in the same configuration. (That is, if there are people in chairs 1, 3, 4, 8, then there are people in chairs 11, 13, 14, 18, and vice versa.) This means that the total number of people in chairs is even.
There are 10 possibilities for the number of chairs occupied among the first 10 chairs (1 to 10), and so 10 possibilities for the total number of chairs occupied (the even numbers from 2 to 20, inclusive).
Thus, there are 10 pairs (n, k) that work when $k = 10$.

Consider $k = 5$.
Here, any people in chairs 1 to 5 move to chairs 6 to 10, any people in chairs 6 to 10 move to chairs 11 to 15, any people in chairs 11 to 15 move to chairs 16 to 20, and any people in chairs 16 to 20 move to chairs 1 to 5.
This means that the four sections containing 5 chairs must contain the same number of people in the same configuration.

This means that the total number of people in chairs is a multiple of 4 and can be 4, 8, 12, 16, or 20, which gives 5 possible values of n.
Thus, there are 5 pairs (n, k) that work when $k = 5$.

If we consider $k = 4$ and $k = 2$, in a similar way, we can determine that there are 4 and 2 possible values for n, respectively.
Thus, there are 4 pairs (n, k) that work when $k = 4$ (namely, $(5, 4)$, $(10, 4)$, $(15, 4)$, $(20, 4)$) and 2 pairs (n, k) that work when $k = 2$ (namely $(10, 2)$ and $(20, 2)$).

Consider $k = 1$.
Here, each person in a chair moves 1 chair along the circle.
If there is a person in chair 1, then they move to chair 2, which means that there must have been a person in chair 2. This person has moved to chair 3, which means that there must have been a person in chair 3, and so on.
Continuing, we see that all 20 chairs must be full, and so $n = 20$ is the only possibility.
Thus, there is 1 pair (n, k) that works when $k = 1$.

Consider $k = 3$, $k = 7$ and $k = 9$.
In each of these cases, $n = 20$ is the only possibility.
If $k = 3$, following a similar argument to that for $k = 1$, the person in chair 1 moves to chair 4, the person in chair 4 moves to chair 7, and so on:

$$1 \to 4 \to 7 \to 10 \to 13 \to 16 \to 19 \to 2 \to 5 \to 8 \to 11 \to 14$$

$$\to 17 \to 20 \to 3 \to 6 \to 9 \to 12 \to 15 \to 18 \to 1$$

The cycle only concludes when all 20 chairs are included, and so if 1 chair is occupied, then all chairs are occupied.
A similar cycle can be constructed for $k = 7$ and $k = 9$ and so in each case only $n = 20$ works.

Consider $k = 6$.
Looking at the cycles as in the previous case,

$$1 \to 7 \to 13 \to 19 \to 5 \to 11 \to 17 \to 3 \to 9 \to 15 \to 1$$

$$2 \to 8 \to 14 \to 20 \to 6 \to 12 \to 18 \to 4 \to 10 \to 16 \to 2$$

Note that starting at any chair other than 1 or 2 gives one of these two cycles. That is, all 20 chairs appear here.
This means that each set of 10 chairs are either all occupied or are all not occupied.
This means that there are two possible values of n, namely 10 or 20.
Thus, there are 2 pairs (n, k) that work when $k = 6$.

Consider $k = 8$.
Here, there are 4 cycles:

$$1 \to 9 \to 17 \to 5 \to 13 \to 1 \qquad 2 \to 10 \to 18 \to 6 \to 14 \to 2$$

$$3 \to 11 \to 19 \to 7 \to 15 \to 3 \qquad 4 \to 12 \to 20 \to 8 \to 16 \to 4$$

and so there are 4 values of n.
Note that starting at any chair other than 1, 2, 3 or 4 gives one of these four cycles. That is, all 20 chairs appear here.
Thus, there are 4 pairs (n, k) that work when $k = 8$.

Finally, we consider $k = 11, 12, 13, 14, 15, 16, 17, 18, 19$.

Moving 11 seats clockwise gives the same result as moving 9 seats counterclockwise. Since the arguments above do not depend on the actual direction, then the number of pairs when $k = 11$ equals the number of pairs when $k = 9$.
Similarly, the numbers of pairs for $k = 12, 13, 14, 15, 16, 17, 18, 19$ equal the number of pairs for $k = 8, 7, 6, 5, 4, 3, 2, 1$, respectively.
This gives the following chart:

Values of k	Number of values of n
20	20
10	10
5, 15	5
4, 8, 12, 16	4
2, 6, 14, 18	2
1, 3, 7, 9, 11, 13, 17, 19	1

Therefore, the number of pairs (n, k) is $20 + 10 + 2 \times 5 + 4 \times 4 + 4 \times 2 + 8 \times 1 = 72$.

ANSWER: 72

Part B

1. (a) We arrange the list in increasing order and obtain $4, 6, 7, 9, 13$.
Therefore, the range of this list is $13 - 4 = 9$.

(b) Removing a from the list, the smallest number remaining is 5 and the largest number remaining is 13.
Since $13 - 5 = 8$ which is smaller than 12 (the actual range), then a must be either the smallest number or the largest number in the list.
If a is the smallest number in the list, then 13 is the largest and so for the range to be 12, we must have $13 - a = 12$ and so $a = 1$. Note that if $a = 1$, then the list is $1, 5, 10, 11, 13$, which has range 12.
If a is the largest number in the list, then 5 is the smallest and so for the range to be 12, we must have $a - 5 = 12$ and so $a = 17$. Note that if $a = 17$, then the list is $5, 10, 11, 13, 17$, which has range 12.
Therefore, the two possible values of a are 1 and 17.

(c) Since $x^2 \geq 0$, then $6 \leq 6 + 2x^2$ and $6 + 2x^2 \leq 6 + 4x^2$ and $6 + 4x^2 \leq 6 + 5x^2$.
In other words, when the given list is arranged so that each term is greater than or equal to the one before it, we obtain

$$6, 6 + 2x^2, 6 + 4x^2, 6 + 5x^2$$

Since the range of this list is 80, then $(6 + 5x^2) - 6 = 80$ and so $5x^2 = 80$ or $x^2 = 16$, which means that $x = \pm 4$.

(d) Since $x > 0$ and $y > 0$, then $0 < x + y$ and $x + y < 3x + y$ and $3x + y < 5x + 3y$.
In other words, when the given list is arranged in increasing order, we obtain

$$0, x + y, 3x + y, 5x + 3y$$

Since the range of this list is 19, then $(5x + 3y) - 0 = 19$ or $5x + 3y = 19$.
Since x and y are positive integers, then it must be the case that $x = 2$ and $y = 3$.
(To see why there are no more solutions, we note that since $y > 0$, then $5x < 19$ which means that $x = 1$, $x = 2$ or $x = 3$. We can check that when $x = 1$ and $x = 3$, the value of y is not an integer.)
Thus, $x = 2$ and $y = 3$.

2. (a) Since there are 11 balls, then there are $11 \times 10 = 110$ ways in which Julio can remove two balls.
Since there are 7 black balls, then there are $7 \times 6 = 42$ ways in which Julio can remove two black balls.
Therefore, the probability that both of the balls that Julio removes are black is $\frac{42}{110}$ which equals $\frac{21}{55}$.

(b) Since there are $g + 6$ balls, then there are $(g + 6)(g + 5)$ ways in which Julio can remove two balls.
Since there are 6 black balls, then there are $6 \times 5 = 30$ ways in which Julio can remove two black balls.
Since the probability that both of the balls that Julio removes are black is $\frac{1}{8}$, then

$\dfrac{30}{(g+6)(g+5)} = \dfrac{1}{8}$ and so $(g+6)(g+5) = 240$.
By trial and error, we can determine that $g = 10$.
Alternatively, we can expand and factor:

$$\begin{aligned}(g+6)(g+5) &= 240\\ g^2+11g+30 &= 240\\ g^2+11g-210 &= 0\\ (g+21)(g-10) &= 0\end{aligned}$$

which means that $g = -21$ or $g = 10$.
Since $g > 0$, then $g = 10$.

(c) Since there are $3x$ balls, then there are $3x(3x-1)$ ways in which Julio can remove two balls.
Since there are $2x$ black balls, then there are $2x(2x-1)$ ways in which Julio can remove two black balls.
Since we are given that the probability that both of the balls that Julio removes are black is $\dfrac{7}{16}$, then $\dfrac{2x(2x-1)}{3x(3x-1)} = \dfrac{7}{16}$.
Since $x \neq 0$, then

$$\begin{aligned}\frac{2(2x-1)}{3(3x-1)} &= \frac{7}{16}\\ 32(2x-1) &= 21(3x-1)\\ 64x-32 &= 63x-21\\ x &= 11\end{aligned}$$

Thus, $x = 11$.

(d) Since there are $r+28$ balls, then there are $(r+28)(r+27)(r+26)$ ways in which Julio can remove three balls.
For Julio to draw two black balls and one gold ball, he could draw "black, black, gold" or "black, gold, black" or "gold, black, black".
The number of ways in which the first possibility can happen is $10 \times 9 \times 18$.
The number of ways in which the second possibility can happen is $10 \times 18 \times 9$.
The number of ways in which the third possibility can happen is $18 \times 10 \times 9$.
In other words, there are $3 \times 10 \times 9 \times 18$ ways in which Julio can draw two black balls and one gold ball.
Since the probability that two of the three balls are black and one is gold is at least $\dfrac{1}{3000}$, then $\dfrac{3 \times 10 \times 9 \times 18}{(r+28)(r+27)(r+26)} > \dfrac{1}{3000}$.
Since $3 \times 10 \times 9 \times 18 = 4860$, then this inequality is equivalent to

$$\frac{4860}{(r+28)(r+27)(r+26)} > \frac{4860}{4860 \times 3000}$$

This is true exactly when $(r+28)(r+27)(r+26) < 4860 \times 3000 = 14\,580\,000$.
As r increases, each of $r+28$ and $r+27$ and $r+26$ increases and is positive, so the product $(r+28)(r+27)(r+26)$ also increases and is positive.

This means that we can find an integer r where the product is less than 14 580 000 and for which the next larger integer value of r makes the product larger than 14 580 000. The first of these integers will be the largest value of r for which the probability is at least $\frac{1}{3000}$.

We note that $\sqrt[3]{14\,580\,000} \approx 244$.

Since $r+28$, $r+27$ and $r+26$ are close in size, we start our search near $r+28=244$.

When $r=216$, we get $(r+28)(r+27)(r+26)=14\,348\,664$.

When $r=217$, we get $(r+28)(r+27)(r+26)=14\,526\,540$.

When $r=218$, we get $(r+28)(r+27)(r+26)=14\,705\,880$.

Therefore, $r=217$ is the largest value of r for which the probability is at least $\frac{1}{3000}$.

3. (a) In Figure 1, $\frac{AE}{AC}=\frac{DE}{BC}$.

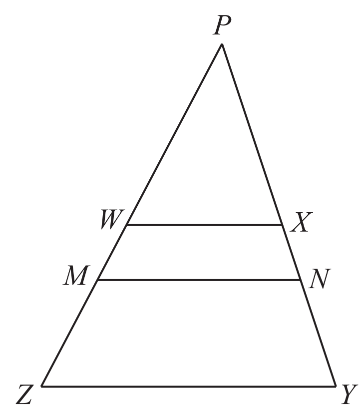
Since $EC=3$ and $AE=x$, then $AC=AE+EC=3+x$.

Thus, $\frac{x}{3+x}=\frac{6}{10}$.

Therefore, $10x=6(3+x)$ or $10x=18+6x$ and so $4x=18$ which gives $x=\frac{9}{2}$.

(b) Since $\frac{WM}{MZ}=\frac{2}{3}$, then $WM=2r$ and $MZ=3r$ for some real number $r>0$.

Extend ZW and YX to meet at point P above WX. (We know that we can do this because $WX<ZY$ which is because $\frac{WX}{ZY}<1$.)



We have now created a new version of the configuration in Figure 1.

Since WX is parallel to ZY, then $\triangle PWX$ and $\triangle PZY$ are similar, which means that $\frac{PW}{PZ}=\frac{WX}{ZY}$.

Using the information determined so far,

$$\begin{aligned}\frac{PW}{PZ} &= \frac{WX}{ZY}\\ \frac{PW}{PW+WM+MZ} &= \frac{3}{4}\\ \frac{PW}{PW+2r+3r} &= \frac{3}{4}\\ 4\times PW &= 3(PW+5r)\\ 4\times PW &= 3\times PW+15r\\ PW &= 15r\end{aligned}$$

We can also note that since WX is parallel to MN, then $\triangle PWX$ and $\triangle PMN$ are similar, which means that $\frac{PW}{PM} = \frac{WX}{MN}$.

Therefore,

$$\frac{WX}{MN} = \frac{PW}{PW+WM} = \frac{15r}{15r+2r} = \frac{15r}{17r}$$

and so $\frac{WX}{MN} = \frac{15}{17}$.

(c) Since $\frac{WX}{ZY} = \frac{3}{4}$, then $WX = 3k$ and $ZY = 4k$ for some real number $k > 0$.

Note that each of $WX = 3k$ and $ZY = 4k$ is an integer.

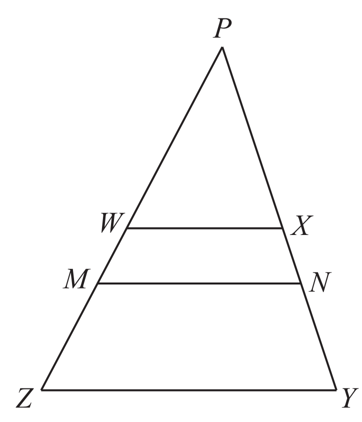
Since $k = ZY - WX$, then k itself is an integer.

Since $\frac{MZ}{WM} = \frac{NY}{XN}$ and this ratio is an integer, then we can set $\frac{MZ}{WM} = t$ for some positive integer t.

If $WM = p$ for some real number $p > 0$, then $MZ = pt$.

As in (b), we extend ZW and YX to meet at P above WX.

We have again created a new version of the configuration in Figure 1.



Since WX is parallel to ZY, then $\triangle PWX$ and $\triangle PZY$ are similar, which means that $\frac{PW}{PZ} = \frac{WX}{ZY}$.

Using the information determined so far,

$$\begin{aligned}\frac{PW}{PZ} &= \frac{WX}{ZY}\\ \frac{PW}{PW+WM+MZ} &= \frac{3}{4}\\ \frac{PW}{PW+p+pt} &= \frac{3}{4}\\ 4\times PW &= 3(PW+p(t+1))\\ 4\times PW &= 3\times PW+3p(t+1)\\ PW &= 3p(t+1)\end{aligned}$$

We can also note that since WX is parallel to MN, then $\triangle PWX$ and $\triangle PMN$ are similar, which means that $\frac{PW}{PM}=\frac{WX}{MN}$.
Since $p\neq 0$, then

$$\frac{WX}{MN}=\frac{PW}{PW+WM}=\frac{3p(t+1)}{3p(t+1)+p}=\frac{p(3t+3)}{p(3t+4)}=\frac{3t+3}{3t+4}$$

Since $WX=3k$, then $\frac{3k}{MN}=\frac{3t+3}{3t+4}$ and so $MN=\frac{(3t+4)3k}{3t+3}=\frac{(3t+4)k}{t+1}$.
Now $MN=\frac{(3t+4)k}{t+1}=\frac{3(t+1)k+k}{t+1}=3k+\frac{k}{t+1}$.
Since MN is an integer and k is an integer, then $\frac{k}{t+1}$ is an integer, which means that the integer k is a multiple of the integer $t+1$, say $k=q(t+1)$ for some positive integer q.
We are given that $WX+MN+ZY=2541$.
Substituting, we obtain successively the equivalent equations

$$\begin{aligned}3k+\left(3k+\frac{k}{t+1}\right)+4k &= 2541\\ 10k+\frac{k}{t+1} &= 2541\\ 10q(t+1)+q &= 2541\\ q(10t+10+1) &= 2541\\ q(10t+11) &= 2541\end{aligned}$$

Now q and t are positive integers.
Since $t\geq 1$, then $10t+11\geq 21$ and $10t+11$ has units digit 1.
So q and $10t+11$ form a positive integer factor pair of 2541 with each factor having units digit 1. (Since 2541 and $10t+11$ have units digit 1, then q must also have units digit 1.)
Factoring 2541, we obtain

$$2541=3\times 847=3\times 7\times 121=3\times 7\times 11^2$$

The positive divisors of 2541 are

$$1,3,7,11,21,33,77,121,231,363,847,2541$$

The divisor pairs consisting of two integers each with units digit 1 are

$$2541 = 1 \times 2541 = 11 \times 231 = 21 \times 121$$

Since $10t + 1 \geq 21$, then we can have

- $q = 1$ and $10t + 11 = 2541$, which gives $t = 253$
- $q = 11$ and $10t + 11 = 231$, which gives $t = 22$
- $q = 21$ and $10t + 11 = 121$, which gives $t = 11$
- $q = 121$ and $10t + 11 = 21$, which gives $t = 1$

We are asked for the possible lengths of

$$MN = 3k + \frac{k}{t+1} = 3q(t+1) + q = q(3t+3+1) = q(3t+4)$$

Using the values of q and t above, we obtain

- $q = 1$ and $t = 253$: $MN = 763$
- $q = 11$ and $t = 22$: $MN = 770$
- $q = 21$ and $t = 11$: $MN = 777$
- $q = 121$ and $t = 1$: $MN = 847$

Therefore, the possible lengths for MN are $763, 770, 777, 847$.

It is indeed possible to construct a trapezoid in each of these cases. The corresponding lengths of WX are $762, 759, 756, 726$, respectively, and the correponding lengths of ZY are $1016, 1012, 1008, 968$, respectively. One way to try to construct these is by putting right angles at W and Z.