The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Pascal Contest

(Grade 9)

Tuesday, February 28, 2017
(in North America and South America)

Wednesday, March 1, 2017
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{4\times 3}{2+1} = \dfrac{12}{3} = 4$.

   Answer: (A)

2. In each of the 6 rows, there is 1 unshaded square, and so there are $6-1=5$ shaded squares per row.
   Since there are 6 rows, then there are $6\times 5 = 30$ shaded squares.
   Alternatively, we note that there are $6\times 6 = 36$ squares in the grid and 6 unshaded squares, which means there are $36-6=30$ shaded squares in the grid.

   Answer: (B)

3. In the diagram, there are 5 shaded and 3 unshaded triangles, and so the ratio of the number of shaded triangles to the number of unshaded triangles is $5:3$.

   Answer: (B)

4. We note that $7=\sqrt{49}$ and that $\sqrt{40} < \sqrt{49} < \sqrt{50} < \sqrt{60} < \sqrt{70} < \sqrt{80}$.
   This means that $\sqrt{40}$ or $\sqrt{50}$ is the closest to 7 of the given choices.
   Since $\sqrt{40}\approx 6.32$ and $\sqrt{50}\approx 7.07$, then $\sqrt{50}$ is closest to 7.

   Answer: (C)

5. We need to determine the time that is 30 hours after 2 p.m. on Friday.
   The time that is 24 hours after 2 p.m. on Friday is 2 p.m. on Saturday.
   The time that is 30 hours after 2 p.m. on Friday is an additional 6 hours later.
   This time is 8 p.m. on Saturday.

   Answer: (E)

6. The time period in which the number of people at the zoo had the largest increase is the time period over which the height of the bars in the graph increases by the largest amount.
   Looking at the graph, this is between 11:00 a.m. and 12:00 p.m.
   (We note that the first three bars represent numbers between 200 and 400, and the last three bars represent numbers between 600 and 800, so the time period between 11:00 a.m. and 12:00 p.m. is the only one in which the increase was larger than 200.)

   Answer: (C)

7. Since $2x-3=10$, then $2x=13$ and so $4x=2(2x)=2(13)=26$.
   (We did not have to determine the value of x.)

   Answer: (D)

8. The three integers from the list whose product is 80 are 1, 4 and 20, since $1\times 4\times 20 = 80$.
   The sum of these integers is $1+4+20=25$.
   (Since 80 is a multiple of 5 and 20 is the only integer in the list that is a multiple of 5, then 20 must be included in the product. This leaves two integers to choose, and their product must be $\dfrac{80}{20}=4$. From the given list, these integers must be 1 and 4.)

   Answer: (C)

9. Since Jovin, Anna and Olivia take $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{4}$ of the pizza, respectively, then the fraction of the pizza with which Wally is left is

$$1-\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{4} = \tfrac{12}{12}-\tfrac{4}{12}-\tfrac{2}{12}-\tfrac{3}{12} = \tfrac{3}{12} = \tfrac{1}{4}\,.$$

   Answer: (B)

10. When $n = 1$, the values of the five expressions are $2014, 2018, 2017, 2018, 2019$.
When $n = 2$, the values of the five expressions are $2011, 2019, 4034, 2021, 2021$.
Only the fifth expression $(2017 + 2n)$ is odd for both of these choices of n, so this must be the correct answer.
We note further that since 2017 is an odd integer and $2n$ is always an even integer, then $2017 + 2n$ is always an odd integer, as required.

Answer: (E)

11. When Ursula runs 30 km at 10 km/h, it takes her $\dfrac{30 \text{ km}}{10 \text{ km/h}} = 3$ h.
This means that Jeff completes the same distance in $3 \text{ h} - 1 \text{ h} = 2$ h.
Therefore, Jeff's constant speed is $\dfrac{30 \text{ km}}{2 \text{ h}} = 15$ km/h.

Answer: (D)

12. Since the area of the larger square equals the sum of the areas of the shaded and unshaded regions inside, then the area of the larger square equals $2 \times 18 \text{ cm}^2 = 36 \text{ cm}^2$.
Since the larger square has an area of 36 cm^2, then its side length is $\sqrt{36 \text{ cm}^2} = 6$ cm.

Answer: (C)

13. *Solution 1*
We undo Janet's steps to find the initial number.
To do this, we start with 28, add 4 (to get 32), then divide the sum by 2 (to get 16), then subtract 7 (to get 9).
Thus, Janet's initial number was 9.

Solution 2
Let Janet's initial number be x.
When she added 7 to her initial number, she obtained $x + 7$.
When she multiplied this sum by 2, she obtained $2(x + 7)$ which equals $2x + 14$.
When she subtracted 4 from this result, she obtained $(2x + 14) - 4$ which equals $2x + 10$.
Since her final result was 28, then $2x + 10 = 28$ or $2x = 18$ and so $x = 9$.

Answer: (A)

14. Since the tax rate is 10%, then the tax on each \$2.00 app is $\$2.00 \times \dfrac{10}{100} = \$0.20$.
Therefore, including tax, each app costs $\$2.00 + \$0.20 = \$2.20$.
Since Tobias spends \$52.80 on apps, he downloads $\dfrac{\$52.80}{\$2.20} = 24$ apps.
Therefore, $m = 24$.

Answer: (D)

15. Let s be the side length of the square with area k.
The sum of the heights of the squares on the right side is $3 + 8 = 11$.
The sum of the heights of the squares on the left side is $1 + s + 4 = s + 5$.
Since the two sums are equal, then $s + 5 = 11$, and so $s = 6$.
Therefore, the square with area k has side length 6, and so its area is $6^2 = 36$.
In other words, $k = 36$.

Answer: (C)

16. The six angles around the centre of the spinner add to 360°.
Thus, $140^\circ + 20^\circ + 4x^\circ = 360^\circ$ or $4x = 360 - 140 - 20 = 200$, and so $x = 50$.
Therefore, the sum of the central angles of the shaded regions is $140^\circ + 50^\circ + 50^\circ = 240^\circ$.
The probability that the spinner lands on a shaded region is the fraction of the entire central angle that is shaded, which equals the sum of the central angles of the shaded regions divided by the total central angle (360°), or $\frac{240^\circ}{360^\circ} = \frac{2}{3}$. (We can ignore the possibility that the spinner lands exactly on one of the dividing lines, since we assume that they are infinitesimally thin.)

ANSWER: (A)

17. Since Igor is shorter than Jie, then Igor cannot be the tallest.
Since Faye is taller than Goa, then Goa cannot be the tallest.
Since Jie is taller than Faye, then Faye cannot be the tallest.
Since Han is shorter than Goa, then Han cannot be the tallest.
The only person of the five who has not been eliminated is Jie, who must thus be the tallest.

ANSWER: (E)

18. From the number line shown, we see that $x < x^3 < x^2$.
If $x > 1$, then successive powers of x are increasing (that is, $x < x^2 < x^3$).
Since this is not the case, then it is not true that $x > 1$.
If $x = 0$ or $x = 1$, then successive powers of x are equal. This is not the case either.
If $0 < x < 1$, then successive powers of x are decreasing (that is, $x^3 < x^2 < x$). This is not the case either.
Therefore, it must be the case that $x < 0$.
If $x < -1$, we would have $x^3 < x < 0 < x^2$. This is because when $x < -1$, then x is negative and we have $x^2 > 1$ which gives $x^3 = x^2 \times x < 1 \times x$. This is not the case here either.
Therefore, it must be the case that $-1 < x < 0$.
From the given possibilities, this means that $-\frac{2}{5}$ is the only possible value of x.
We can check that if $x = -\frac{2}{5} = -0.4$, then $x^2 = 0.16$ and $x^3 = -0.064$, and so we have $x < x^3 < x^2$. We can also check by substitution that none of the other possible answers gives the correct ordering of x, x^2 and x^3.

ANSWER: (C)

19. Since $\angle XMZ = 30^\circ$, then $\angle XMY = 180^\circ - \angle XMZ = 180^\circ - 30^\circ = 150^\circ$.
Since the angles in $\triangle XMY$ add to 180°, then

$$\angle YXM = 180^\circ - \angle XYZ - \angle XMY = 180^\circ - 15^\circ - 150^\circ = 15^\circ$$

(Alternatively, since $\angle XMZ$ is an exterior angle of $\triangle XMY$, then $\angle XMZ = \angle YXM + \angle XYM$ which also gives $\angle YXM = 15^\circ$.)
Since $\angle XYM = \angle YXM$, then $\triangle XMY$ is isosceles with $MX = MY$.
But M is the midpoint of YZ, and so $MY = MZ$.
Since $MX = MY$ and $MY = MZ$, then $MX = MZ$.
This means that $\triangle XMZ$ is isosceles with $\angle XZM = \angle ZXM$.
Therefore, $\angle XZY = \angle XZM = \frac{1}{2}(180^\circ - \angle XMZ) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.

ANSWER: (A)

20. We call the $n \times n \times n$ cube the "large cube", and we call the $1 \times 1 \times 1$ cubes "unit cubes". The unit cubes that have exactly 0 gold faces are those unit cubes that are on the "inside" of the large cube.
In other words, these are the unit cubes none of whose faces form a part of any of the faces of the large cube.
These unit cubes form a cube that is $(n-2) \times (n-2) \times (n-2)$.
To see why this is true, imagine placing the original painted large cube on a table.
Each unit cube with at least one face that forms part of one of the outer faces (or outer layers) has paint on at least one face.
First, we remove the top and bottom layers of unit cubes. This creates a rectangular prism that is $n-2$ cubes high and still has a base that is $n \times n$.
Next, we can remove the left, right, front, and back faces.
This leaves a cube that is $(n-2) \times (n-2) \times (n-2)$.
Therefore, $(n-2)^3$ unit cubes have 0 gold faces.

The unit cubes that have exactly 1 gold face are those unit cubes that are on the outer faces of the large cube but do not touch the edges of the large cube.
Consider each of the six $n \times n$ faces of the large cube. Each is made up of n^2 unit cubes.
The unit cubes that have 1 gold face are those with at least one face that forms part of a face of the large cube, but do not share any edges with the edges of the large cube. Using a similar argument to above, we can see that these unit cubes form a $(n-2) \times (n-2)$ square.
There are thus $(n-2)^2$ cubes on each of the 6 faces that have 1 painted face, and so $6(n-2)^2$ cubes with 1 painted face.
We calculate the values of $(n-2)^3$ and $6(n-2)^2$ for each of the possible choices for n:

Choice	n	$(n-2)^3$	$6(n-2)^2$
(A)	7	125	150
(B)	8	216	216
(C)	9	343	294
(D)	10	512	384
(E)	4	8	24

From this information, the smallest possible value of n when $(n-2)^3$ is larger than $6(n-2)^2$ must be $n = 9$.

To see this in another way, we can ask the question "When is $(n-2)^3$ greater than $6(n-2)^2$?". Note that $(n-2)^3 = (n-2) \times (n-2)^2$ and $6(n-2)^2 = 6 \times (n-2)^2$, and so $(n-2)^3$ is greater than $6(n-2)^2$ when $(n-2)$ is greater than 6, which is when n is greater than 8.
The smallest positive integer value of n for which this is true is $n = 9$.

Answer: (C)

21. The averages of groups of three numbers are equal if the sums of the numbers in each group are equal, because in each case the average equals the sum of the three numbers divided by 3. Therefore, the averages of three groups of three numbers are equal if the sum of each of the three groups are equal.
The original nine numbers have a sum of

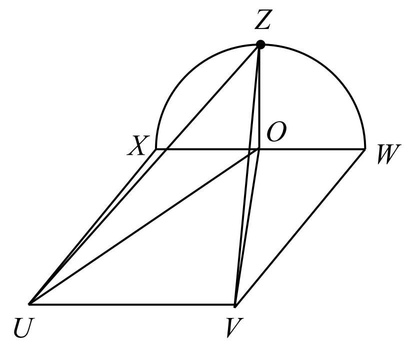
$$1 + 5 + 6 + 7 + 13 + 14 + 17 + 22 + 26 = 111$$

and so if these are divided into three groups of equal sum, the sum of each group is $\frac{111}{3} = 37$.
Consider the middle three numbers. Since two of the numbers are 13 and 17, then the third

number must be $37 - 13 - 17 = 7$. We note that the remaining six numbers can be split into the groups $5, 6, 26$ and $1, 14, 22$, each of which also has a sum of 37.
Therefore, the number that is placed in the shaded circle is 7.

ANSWER: (D)

22. The perimeter of $\triangle UVZ$ equals $UV + UZ + VZ$.
We know that $UV = 20$. We need to calculate UZ and VZ.
Let O be the point on XW directly underneath Z.
Since Z is the highest point on the semi-circle and XW is the diameter, then O is the centre of the semi-circle.
We join UO, VO, UZ, and VZ.



Since $UVWX$ is a rectangle, then $XW = UV = 20$ and $UX = VW = 30$.
Since XW is a diameter of the semi-circle and O is the centre, then O is the midpoint of XW and so $XO = WO = 10$.
This means that the radius of the semi-circle is 10, and so $OZ = 10$ as well.
Now $\triangle UXO$ and $\triangle VWO$ are both right-angled, since $UVWX$ is a rectangle.
By the Pythagorean Theorem, $UO^2 = UX^2 + XO^2 = 30^2 + 10^2 = 900 + 100 = 1000$ and $VO^2 = VW^2 + WO^2 = 30^2 + 10^2 = 1000$.
Each of $\triangle UOZ$ and $\triangle VOZ$ is right-angled at O, since the semi-circle is vertical and the rectangle is horizontal.
Therefore, we can apply the Pythagorean Theorem again to obtain $UZ^2 = UO^2 + OZ^2$ and $VZ^2 = VO^2 + OZ^2$.
Since $UO^2 = VO^2 = 1000$, then $UZ^2 = VZ^2 = 1000 + 10^2 = 1100$ or $UZ = VZ = \sqrt{1100}$.
Therefore, the perimeter of $\triangle UVZ$ is $20 + 2\sqrt{1100} \approx 86.332$.
Of the given choices, this is closest to 86.

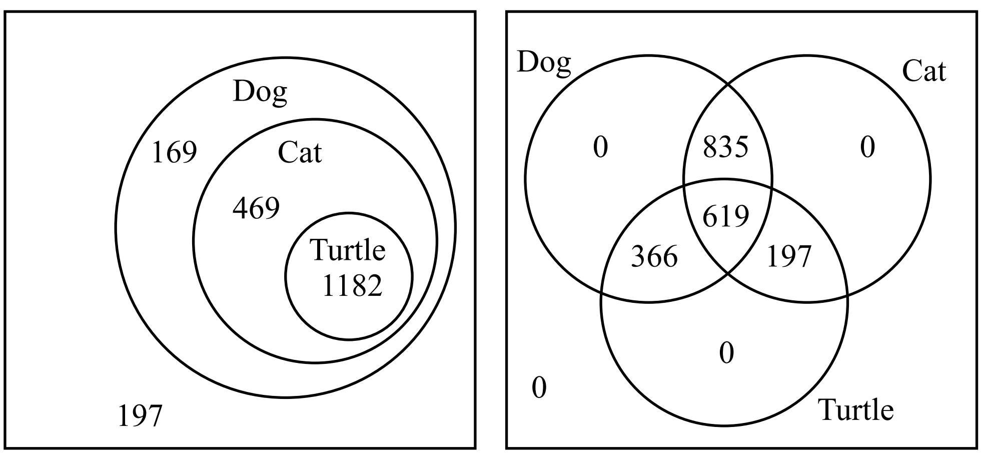
ANSWER: (B)

23. The squares of the one-digit positive integers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are $1, 4, 9, 16, 25, 36, 49, 64, 81$, respectively.
Of these, the squares $1, 25, 36$ end with the digit of their square root.
In other words, $k = 1, 5, 6$ are Anderson numbers.
Thus, $k = 6$ is the only even one-digit Anderson number.
To find all even two-digit Anderson numbers, we note that any two-digit even Anderson number k must have a units (ones) digit of 6. This is because the units digit of k and the units digit of k^2 must match (by the definition of an Anderson number) and because the units digit of k completely determines the units digit of k^2. (We can see this by doing "long multiplication".)
So we need to look for two-digit Anderson numbers k with digits $c6$.
Another way of writing the number $c6$ is $k = 10c+6$. (This form uses the place values associated with the digits.)
In this case, $k^2 = (10c+6)^2 = (10c+6)(10c+6) = (10c)^2+6(10c)+10c(6)+6^2 = 100c^2+120c+36$.
Note that $k^2 = 100(c^2 + c) + 10(2c + 3) + 6$ and so the units digit of k^2 is 6.
For k to be an Anderson number, we need the tens digit of k^2 to be c, in which case the final two digits of k^2 will be $c6$.
Thus, the tens digit of k^2 is equal to the units digit of $2c + 3$.
This means that $k = 10c + 6$ is an Anderson number exactly when the units digit of $2c + 3$ is equal to the digit c.
When we check the nine possible values for c, we find that the only possibility is that $c = 7$.
This means that $k = 76$ is the only two-digit even Anderson number.
Note that $76^2 = 5776$, which ends with the digits 76.
Next, we look for three-digit even Anderson numbers k.
Using a similar argument to above, we see that k must have digits $b76$.
In other words, $k = 100b + 76$ for some digit b.
In this case, $k^2 = (100b + 76)^2 = 10000b^2 + 15200b + 5776$.
We note that the tens and units digits of k^2 are 76, which means that, for k to be an Anderson number, the hundreds digit of k^2 must be b.
Now $k^2 = 1000(10b^2 + 15b + 5) + 100(2b + 7) + 76$.
Thus, k is an Anderson number exactly when the units digit of $2b + 7$ is equal to the digit b.
Again, checking the nine possible values for b shows us that $b = 3$ is the only possibility.
This means that $k = 376$ is the only three-digit even Anderson number.
Note that $376^2 = 141\,376$, which ends with the digits 376.
Since Anderson numbers are less than 10 000, then we still need to look for four-digit even Anderson numbers.
Again, using a similar argument, we see that k must have digits $a376$.
In other words, $k = 1000a + 376$ for some digit a.
In this case, $k^2 = (1000a + 376)^2 = 1\,000\,000a^2 + 752000a + 141\,376$.
We note that the hundreds, tens and units digits of k^2 are 376, which means that, for k to be an Anderson number, the thousands digit of k^2 must be a.
Now $k^2 = 10000(100a^2 + 75a + 14) + 1000(2a + 1) + 376$.
Thus, k is an Anderson number exactly when the units digit of $2a + 1$ is equal to the digit a.
Again, checking the nine possible values for a shows us that $a = 9$ is the only possibility.
This means that $k = 9376$ is the only four-digit even Anderson number.
Note that $9376^2 = 87\,909\,376$, which ends with the digits 9376.
Thus, S, the sum of the even Anderson numbers, equals $6 + 76 + 376 + 9376 = 9834$.
The sum of the digits of S is $9 + 8 + 3 + 4 = 24$.

ANSWER: (E)

24. Since there are 1182 houses that have a turtle, then there cannot be more than 1182 houses that have a dog, a cat, and a turtle.
Since there are more houses with dogs and more houses with cats than there are with turtles, it is possible that all 1182 houses that have a turtle also have a dog and a cat.
Therefore, the maximum possible number of houses that have all three animals is 1182, and so $x = 1182$.

Since there are 1182 houses that have a turtle and there are 2017 houses in total, then there are $2017 - 1182 = 835$ houses that do not have a turtle.
Now, there are 1651 houses that have a cat.
Since there are 835 houses that do not have a turtle, then there are at most 835 houses that have a cat and do not have a turtle. In other words, not all of the houses that do not have a turtle necessarily have a cat.
This means that there are at least $1651 - 835 = 816$ houses that have both a cat and a turtle.
Lastly, there are 1820 houses that have a dog.
Since there are at least 816 houses that have both a cat and a turtle, then there are at most $2017 - 816 = 1201$ houses that either do not have a cat or do not have a turtle (or both).
Since there are 1820 houses that do have a dog, then there are at least $1820 - 1201 = 619$ houses that have a dog and have both a cat and a turtle as well.
In other words, the minimum possible number of houses that have all three animals is 619, and so $y = 619$.
The two Venn diagrams below show that each of these situations is actually possible:



Since $x = 1182$ and $y = 619$, then $x - y = 563$.

Answer: (C)

25. We label the digits of the unknown number as $vwxyz$.
Since $vwxyz$ and 71794 have 0 matching digits, then $v \neq 7$ and $w \neq 1$ and $x \neq 7$ and $y \neq 9$ and $z \neq 4$.
Since $vwxyz$ and 71744 have 1 matching digit, then the preceding information tells us that $y = 4$.
Since $vwx4z$ and 51545 have 2 matching digits and $w \neq 1$, then $vwxyz$ is of one of the following three forms: $5wx4z$ or $vw54z$ or $vwx45$.

Case 1: $vwxyz = 5wx4z$
Since $5wx4z$ and 21531 have 1 matching digit and $w \neq 1$, then either $x = 5$ or $z = 1$.
If $x = 5$, then $5wx4z$ and 51545 would have 3 matching digits, which violates the given condition. Thus, $z = 1$.

Thus, $vwxyz = 5wx41$ and we know that $w \neq 1$ and $x \neq 5, 7$.
To this point, this form is consistent with the 1st, 2nd, 3rd and 7th rows of the table.
Since $5wx41$ and 59135 have 1 matching digit, this is taken care of by the fact that $v = 5$ and we note that $w \neq 9$ and $x \neq 1$.
Since $5wx41$ and 58342 have 2 matching digits, this is taken care of by the fact that $v = 5$ and $y = 4$, and we note that $w \neq 8$ and $x \neq 3$.
Since $5wx41$ and 37348 have 2 matching digits and $y = 4$, then either $w = 7$ or $x = 3$.
But we already know that $x \neq 3$, and so $w = 7$.
Therefore, $vwxyz = 57x41$ with the restrictions that $x \neq 1, 3, 5, 7$.
We note that the integers $57041, 57241, 57441, 57641, 57841, 57941$ satisfy the requirements, so are all possibilities for Sam's numbers.

Case 2: $vwxyz = vw54z$
Since $vw54z$ and 51545 have only 2 matching digits, so $v \neq 5$ and $z \neq 5$.
Since $vw54z$ and 21531 have 1 matching digit, then this is taken care of by the fact that $x = 5$, and we note that $v \neq 2$ and $z \neq 1$. (We already know that $w \neq 1$.)
Since $vw54z$ and 59135 have 1 matching digit, then $v = 5$ or $w = 9$ or $z = 5$.
This means that we must have $w = 9$.
Thus, $vwxyz = v954z$ and we know that $v \neq 2, 7, 5$ and $z \neq 1, 4, 5$.
To this point, this form is consistent with the 1st, 2nd, 3rd, 4th, and 7th rows of the table.
Since $v954z$ and 58342 have 2 matching digits and $v \neq 5$, then $z = 2$.
Since $v9542$ and 37348 have 2 matching digits, then $v = 3$.
In this case, the integer 39542 is the only possibility, and it satisfies all of the requirements.

Case 3: $vwxyz = vwx45$
Since $vwx45$ and 21531 have 1 matching digit and we know that $w \neq 1$, then $v = 2$ or $x = 5$.
But if $x = 5$, then $vw545$ and 51545 would have 3 matching digits, so $x \neq 5$ and $v = 2$.
Thus, $vwxyz = 2wx45$ and we know that $w \neq 1$ and $x \neq 5, 7$.
To this point, this form is consistent with the 1st, 2nd, 3rd and 7th rows of the table.
Since $2wx45$ and 59135 have 1 matching digit, this is taken care of by the fact that $z = 5$ and we note that $w \neq 9$ and $x \neq 1$.
Since $2wx45$ and 58342 have 2 matching digits, then $w = 8$ or $x = 3$, but not both.
Since $2wx45$ and 37348 have 2 matching digits, then $w = 7$ or $x = 3$, but not both.
If $w = 8$, then we have to have $x \neq 3$, and so neither $w = 7$ nor $x = 3$ is true.
Thus, it must be the case that $x = 3$ and $w \neq 7, 8$.
Therefore, $vwxyz = 2w345$ with the restrictions that $w \neq 1, 7, 8, 9$.
We note that the integers $20345, 22345, 23345, 24345, 25345, 26345$ satisfy the requirements, so are all possibilities for Sam's numbers.

Thus, there are 13 possibilities for Sam's numbers and the sum of these is 526758.

ANSWER: (E)