The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Galois Contest

Wednesday, April 12, 2017
(in North America and South America)

Thursday, April 13, 2017
(outside of North America and South America)

Solutions

1. (a) In Box E, 6 of the 30 cups were purple.
   The percentage of purple cups in Box E was $\frac{6}{30} \times 100\% = \frac{2}{10} \times 100\% = 20\%$.

   (b) On Monday, 30% of Daniel's 90 cups or $30\% \times 90 = \frac{30}{100} \times 90 = 27$ cups were purple.
   Daniel had 9 purple cups in Box D and 6 purple cups in Box E.
   Therefore, the number of purple cups in Box F was $27 - 9 - 6 = 12$.

   (c) Daniel had 27 purple cups and 90 cups in total.
   On Tuesday, Avril added 9 more purple cups to Daniel's cups, bringing the number of purple cups to $27 + 9 = 36$, and the total number of cups to $90 + 9 = 99$.
   Barry brought some yellow cups and included them with the 99 cups.
   Let the number of yellow cups that Barry brought be y.
   The total number of cups was then $99 + y$, while the number of purple cups was still 36 (since Barry brought yellow cups only).
   Since the percentage of cups that were purple was again 30% or $\frac{30}{100}$, then $\frac{30}{100}$ of $99 + y$ must equal 36.
   Solving, we get $\frac{30}{100} \times (99 + y) = 36$ or $30(99 + y) = 3600$ or $99 + y = 120$, and so $y = 21$.
   Therefore, Barry brought 21 cups.

2. (a) Abdi arrived at 5:02 a.m., and so Abdi paid \$5.02.
   Caleigh arrived at 5:10 a.m., and so Caleigh paid \$5.10.
   In total, Abdi and Caleigh paid \$5.02 + \$5.10 = \$10.12.

   (b) If both Daniel and Emily had arrived at the same time, then they each would have paid the same amount, or \$12.34 ÷ 2 = \$6.17.
   In this case, they would have both arrived at 6:17 a.m.
   If Daniel arrived 5 minutes earlier, at 6:12 a.m., and Emily arrived 5 minutes later, at 6:22 a.m., then they would have arrived 10 minutes apart and in total they would have still paid \$12.34.
   (We may check that these arrival times are 10 minutes apart, and that Daniel and Emily's total price is \$6.12 + \$6.22 = \$12.34, as required.)

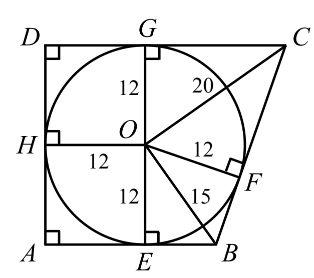
   (c) To minimize the amount that Karla could have paid, we maximize the amount that Isaac and Jacob pay.
   Isaac and Jacob arrived together and Karla arrived after.
   Since Karla arrived at a later time than Isaac and Jacob, then Karla paid more than Isaac and Jacob.
   If Isaac and Jacob both arrived together at 6:18 a.m., then they would each have paid \$6.18, and Karla would have paid \$18.55 − \$6.18 − \$6.18 = \$6.19.
   This is the minimum amount that Karla could have paid. Why?
   If Isaac and Jacob arrived at 6:19 a.m. or later, then Karla would have arrived at a time earlier than 6:19 a.m. (since \$18.55 − \$6.19 − \$6.19 = \$6.17).
   Since Karla arrived after Isaac and Jacob, this is not possible.
   If Isaac and Jacob arrived earlier than 6:18 a.m., then they would have each paid less than \$6.18, and so Karla would have paid more than \$6.19.
   Therefore, the minimum amount that Karla could have paid is \$6.19.

   (d) If Larry arrived earlier than 5:39 a.m., then he would have paid less than \$5.39 and so Mio would have paid more than \$11.98 − \$5.39 = \$6.59.
   Since Mio arrived during the time of the special pricing, it is not possible for Mio to have

paid more than \$6.59, and so Larry must have arrived at 5:39 a.m. or later.
If Larry arrived between 5:39 a.m. and 5:59 a.m. (inclusive), then Larry would have paid the amount between \$5.39 and \$5.59 corresponding to his arrival time.
Therefore, Mio would have paid an amount between \$11.98 − \$5.59 = \$6.39 and \$11.98 − \$5.39 = \$6.59 (inclusive).
Each of the amounts between \$6.39 and \$6.59 corresponds to an arrival time for Mio between 6:39 a.m. and 6:59 a.m., each of which is a possible time that Mio could have arrived during the special pricing period.
That is, each arrival time for Larry from 5:39 a.m. to 5:59 a.m. corresponds to an arrival time for Mio from 6:39 a.m. to 6:59 a.m.
Each of these times is during the period of the special pricing and each corresponding pair of times gives a total price of \$11.98.
To see this, consider that if Larry arrived x minutes after 5:39 a.m. (where x is an integer and $0 \leq x \leq 20$), then Mio arrived x minutes before 6:59 a.m., and in total they paid $\$5.39 + x¢ + \$6.59 - x¢ = \$11.98$.
Since Larry's arrival time and Mio's arrival time may be switched to give the same total, \$11.98, then Larry could also have arrived between 6:39 a.m. and 6:59 a.m.
The only times left to consider are those from 6:00 a.m. to 6:38 a.m.
If Larry arrived at one of these times, his price would have been between \$6.00 and \$6.38, and so Mio's price would have been between \$11.98 − \$6.38 = \$5.60 and \$11.98 − \$6.00 = \$5.98.
Since there are no arrival times which correspond to Mio having to pay an amount between \$5.60 and \$5.98, then it is not possible that Larry arrived at any time from 6:00 a.m. to 6:38 a.m.
Therefore, the ranges of times during which Larry could have arrived are 5:39 a.m. to 5:59 a.m or 6:39 a.m. to 6:59 a.m.

3. (a) Since $\angle OPQ = 90^\circ$, then $\triangle OPQ$ is a right-angled triangle.
By the Pythagorean Theorem, $OQ^2 = OP^2 + PQ^2 = 18^2 + 24^2 = 900$, and so $OQ = \sqrt{900} = 30$ (since $OQ > 0$).
Line segment OS is a radius of the circle and thus has length 18.
Therefore, $SQ = OQ - OS = 30 - 18 = 12$.

(b) Sides AB, BC, CD, and DA are tangent to the circle at points E, F, G, and H, respectively.
Therefore, radii OE, OF, OG, and OH are perpendicular to their corresponding sides, as shown.
In quadrilateral $DHOG$, $\angle OGD = \angle GDH = \angle DHO = 90^\circ$ and so $\angle GOH = 90^\circ$.
Since $OH = OG = 12$ (they are radii of the circle), then $DHOG$ is a square with side length 12.

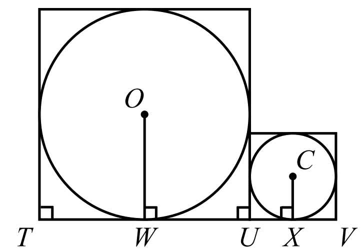


Similarly, $HAEO$ is also a square with side length 12.
Since $\angle OGC = 90^\circ$, then $\triangle OGC$ is a right-angled triangle.
By the Pythagorean Theorem, $GC^2 = OC^2 - OG^2 = 20^2 - 12^2 = 256$, and so $GC = \sqrt{256} = 16$ (since $GC > 0$).
It can be similarly shown that $FC = 16$.
Since $\angle OEB = 90^\circ$, then $\triangle OEB$ is a right-angled triangle.
By the Pythagorean Theorem, $EB^2 = OB^2 - OE^2 = 15^2 - 12^2 = 81$, and so $EB = \sqrt{81} = 9$ (since $EB > 0$).
It can be similarly shown that $FB = 9$.

Therefore, the perimeter of $ABCD$ is $GD + DH + HA + AE + EB + BF + FC + CG$ or $4 \times 12 + 2 \times 9 + 2 \times 16 = 98$.

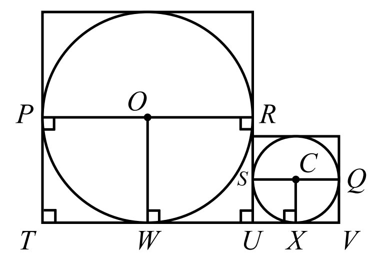
(c) In Figure 1:
Since the circles are inscribed in their respective squares, then TU is a tangent to the larger circle and UV is a tangent to the smaller circle.
Let TU touch the larger circle at W, and let UV touch the smaller circle at X.
The radius OW is perpendicular to TU, and the radius CX is perpendicular to UV.

Figure 1



In Figure 2:
The diameter of the larger circle is equal to the side length of the larger square.
To see this, label the points P and R where the vertical sides of the larger square touch the larger circle.
Join P to O and join R to O.
The radius OP is perpendicular to PT and the radius OR is perpendicular to RU.

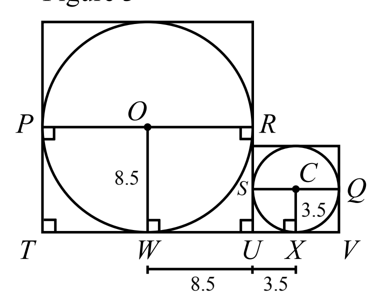
Figure 2



In quadrilateral $PTWO$, $\angle OPT = \angle PTW = \angle TWO = 90^\circ$, and so $\angle POW = 90^\circ$.
Similarly, in quadrilateral $RUWO$, $\angle ROW = 90^\circ$.
Therefore, $\angle POW + \angle ROW = 180^\circ$ and so PR passes through O and is thus a diameter of the larger circle.
In quadrilateral $PTUR$, all 4 interior angles measure 90°, and so $PTUR$ is a rectangle.
It can similarly be shown that if S and Q are the points where the vertical sides of the smaller square touch the smaller circle, then SQ is a diameter of the smaller circle and $SUVQ$ is a rectangle.

In Figure 3:
The area of the larger square is 289, and so each side of the larger square has length $\sqrt{289} = 17$.
The diameter of the larger circle is equal to the side length of the larger square, or $PR = TU = 17$.
Since O is the midpoint of PR, and OW is perpendicular to TU, then W is the midpoint of TU.
Therefore, $WU = OR = OW = 17 \div 2 = 8.5$.
The area of the smaller square is 49, and so each side of the smaller square has length $\sqrt{49} = 7$.

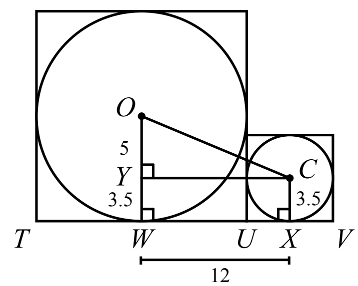
Figure 3



Similarly, X is the midpoint of UV and so $UX = SC = CX = 7 \div 2 = 3.5$.

In Figure 4:
Finally, we construct the line segment from C, parallel to XW, and meeting OW at Y.
In quadrilateral $YWXC$, CY is parallel to XW, YW is perpendicular to XW, and CX is perpendicular to XW, and so $YWXC$ is a rectangle.
Thus, $CX = YW = 3.5$, and
$CY = XW = XU + WU = 3.5 + 8.5 = 12$.
Since $\angle OYC = 90^\circ$, then $\triangle OYC$ is a right-angled triangle with $CY = 12$, and $OY = OW - YW = 8.5 - 3.5 = 5$.

Figure 4



By the Pythagorean Theorem, $OC^2 = CY^2 + OY^2 = 12^2 + 5^2 = 144 + 25 = 169$, and so $OC = \sqrt{169} = 13$ (since $OC > 0$).

4. (a) The total area of the $m = 14$ by $n = 10$ Koeller-rectangle is $m \times n = 14 \times 10 = 140$.
The dimensions of the shaded area inside a Koeller-rectangle are $(m - 2)$ by $(n - 2)$ since the 1 by 1 squares along the sides are unshaded, so each dimension is reduced by 2.
Therefore, the shaded area of a 14 by 10 Koeller-rectangle is $(14-2)\times(10-2) = 12\times 8 = 96$.
The unshaded area is the difference between the total area and the shaded area, or $mn-(m-2)(n-2) = mn-(mn-2m-2n+4) = 2m+2n-4$ or $2\times 14+2\times 10-4 = 44$.
Finally, r is the ratio of the shaded area to the unshaded area, or $\frac{96}{44} = \frac{24}{11}$ (or $24 : 11$).

(b) As we saw in part (a), the shaded area of an m by n Koeller-rectangle is $(m - 2)(n - 2)$, and the unshaded area is $2m + 2n - 4$.
Therefore, $r = \frac{(m-2)(n-2)}{2m+2n-4}$. When $n = 4$, $r = \frac{2(m-2)}{2m+4} = \frac{2(m-2)}{2(m+2)} = \frac{m-2}{m+2}$.
We rewrite $\frac{m-2}{m+2}$ as $\frac{m+2-4}{m+2} = \frac{m+2}{m+2} - \frac{4}{m+2} = 1 - \frac{4}{m+2}$.
We must determine all possible integer values of u for which $r = 1 - \frac{4}{m+2} = \frac{u}{77}$, for some integer $m \geq 3$.
Simplifying this equation, we get

$$\begin{aligned} 1 - \frac{4}{m+2} &= \frac{u}{77} \\ 1 - \frac{u}{77} &= \frac{4}{m+2} \\ \frac{77-u}{77} &= \frac{4}{m+2} \\ (m+2)(77-u) &= 4 \times 77 \end{aligned}$$

Both u and m are integers, and so $(m + 2)(77 - u)$ is the product of two integers.
If a and b are positive integers so that $ab = 4 \times 77 = 2^2 \times 7 \times 11$, then there are 6 possible factor pairs (a, b) with $a < b$.
These are: $(1, 308), (2, 154), (4, 77), (7, 44), (11, 28)$, and $(14, 22)$.
Since $m \geq 3$, then $m + 2 \geq 5$ and so $m + 2$ cannot equal $1, 2$ and 4.
However, $m+2$ can equal any of the remaining 9 divisors: $7, 11, 14, 22, 28, 44, 77, 154, 308$.
In the table below, we determine the possible values for u given that $(m + 2)(77 - u) = 2^2 \times 7 \times 11$, and $m + 2 \geq 5$.

$m+2$	7	11	14	22	28	44	77	154	308
$77-u$	44	28	22	14	11	7	4	2	1
u	33	49	55	63	66	70	73	75	76

The integer values of u for which there exists a Koeller-rectangle with $n=4$ and $r=\dfrac{u}{77}$, are $u=33,49,55,63,66,70,73,75,76$.

(For example, the 5 by 4 Koeller-rectangle has $r=\dfrac{m-2}{m+2}=\dfrac{3}{7}=\dfrac{33}{77}$, and so $u=33$.)

(c) As in part (b), $r=\dfrac{(m-2)(n-2)}{2m+2n-4}$, and when $n=10$, $r=\dfrac{8(m-2)}{2m+16}=\dfrac{4(m-2)}{m+8}$.

Rearranging this equation, we get

$$\begin{aligned}
r &= \frac{4(m-2)}{m+8}\\
\frac{r}{4} &= \frac{m-2}{m+8}\\
\frac{r}{4} &= \frac{m+8-10}{m+8}\\
\frac{r}{4} &= \frac{m+8}{m+8}-\frac{10}{m+8}\\
\frac{r}{4} &= 1-\frac{10}{m+8}\\
\frac{10}{m+8} &= 1-\frac{r}{4}\\
\frac{10}{m+8} &= 1-\frac{u}{4p^2} \quad (\text{since } r=\frac{u}{p^2})\\
\frac{10}{m+8} &= \frac{4p^2-u}{4p^2}\\
40p^2 &= (m+8)(4p^2-u)
\end{aligned}$$

Since p,u and m are integers, then $(m+8)(4p^2-u)$ is the product of two integers.

We must determine all prime numbers p for which there are exactly 17 positive integer values of u for Koeller-rectangles satisfying this equation $40p^2=(m+8)(4p^2-u)$.

For $p=2,3,5,7$, and then $p\geq 11$, we proceed with the following strategy:

- determine the value of $40p^2$
- count the number of divisors of $40p^2$
- eliminate possible values of $m+8$, thus eliminating possible values of $4p^2-u$
- count the number of values of u by counting the number of values of $4p^2-u$

If $p=2$, then $40p^2=40\times 2^2=2^5\times 5$, and so $2^5\times 5=(m+8)(16-u)$.

Each divisor of $2^5\times 5$ is of the form $2^i\times 5^j$, for integers $0\leq i\leq 5$ and $0\leq j\leq 1$.

That is, there are 6 choices for i (each of the integers from 0 to 5) and 2 choices for j (0 or 1), and so there are $6\times 2=12$ different divisors of $2^5\times 5$.

Since $2^5\times 5=(m+8)(16-u)$, then there are at most 12 different integer values of $16-u$ (the 12 divisors of $2^5\times 5$), and so there are at most 12 different integer values of u.

Therefore, when $p=2$, there cannot be exactly 17 positive integer values of u.

If $p=5$, then $40p^2=40\times 5^2=2^3\times 5^3$, and so $2^3\times 5^3=(m+8)(100-u)$.

Each divisor of $2^3\times 5^3$ is of the form $2^i\times 5^j$, for integers $0\leq i\leq 3$ and $0\leq j\leq 3$.

That is, there are 4 choices for i and 4 choices for j, and so there are $4\times 4=16$ different divisors of $2^3\times 5^3$.

Since $2^3\times 5^3=(m+8)(100-u)$, then there are at most 16 different integer values of

$100 - u$, and so there are at most 16 different integer values of u.
Therefore, when $p = 5$, there cannot be exactly 17 positive integer values of u.

If $p = 3$, then $40p^2 = 40 \times 3^2 = 2^3 \times 3^2 \times 5$, and so $2^3 \times 3^2 \times 5 = (m + 8)(36 - u)$.
Each divisor of $2^3 \times 3^2 \times 5$ is of the form $2^i \times 3^j \times 5^k$, for integers $0 \leq i \leq 3$, $0 \leq j \leq 2$, and $0 \leq k \leq 1$.
That is, there are $4 \times 3 \times 2 = 24$ different divisors of $2^3 \times 3^2 \times 5$.
Since $m \geq 3$, then $m+8 \geq 11$, and so the divisors of $2^3 \times 3^2 \times 5$ which $m+8$ cannot equal are: $1, 2, 3, 4, 5, 6, 8, 9$, and 10.
Since there are 9 divisors which $m + 8$ cannot equal, then there are 9 divisors that $36 - u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 3^2 \times 5$ by each of the 9 divisors $1, 2, 3, 4, 5, 6, 8, 9$, and 10.)
So then there are $24 - 9 = 15$ different integer values of $36 - u$, and so there are exactly 15 different integer values of u when $p = 3$.
Therefore, there are not 17 positive integer values of u when $p = 3$.

If $p = 7$, then $40p^2 = 40 \times 7^2 = 2^3 \times 5 \times 7^2$, and so $2^3 \times 5 \times 7^2 = (m + 8)(196 - u)$.
Each divisor of $2^3 \times 5 \times 7^2$ is of the form $2^i \times 5^j \times 7^k$, for integers $0 \leq i \leq 3$, $0 \leq j \leq 1$, and $0 \leq k \leq 2$.
That is, there are $4 \times 2 \times 3 = 24$ different divisors of $2^3 \times 5 \times 7^2$.
Since $m+8 \geq 11$, then the divisors of $2^3 \times 5 \times 7^2$ that $m+8$ cannot equal are: $1, 2, 4, 5, 7, 8$, and 10.
Since there are 7 divisors which $m + 8$ cannot equal, then there are 7 divisors that $196 - u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 5 \times 7^2$ by each of the 7 divisors $1, 2, 4, 5, 7, 8$, and 10. We also note that each of the remaining divisors that $196 - u$ can equal, is less than 196, giving a positive integer value for u.)
So then there are $24 - 7 = 17$ different integer values of $196 - u$, and so there are exactly 17 different integer values of u when $p = 7$.

For all remaining primes $p \geq 11$, we get $40p^2 = 2^3 \times 5 \times p^2$, and so $2^3 \times 5 \times p^2 = (m + 8)(4p^2 - u)$.
Since $p \neq 2$ and $p \neq 5$, each divisor of $2^3 \times 5 \times p^2$ is of the form $2^i \times 5^j \times p^k$, for integers $0 \leq i \leq 3$, $0 \leq j \leq 1$, and $0 \leq k \leq 2$.
That is, there are $4 \times 2 \times 3 = 24$ different divisors of $2^3 \times 5 \times p^2$.
Since $m+8 \geq 11$ and $p \geq 11$, then the divisors of $2^3 \times 5 \times p^2$ that $m+8$ cannot equal are: $1, 2, 4, 5, 8$, and 10.
Since there are 6 divisors which $m+8$ cannot equal, then there are 6 divisors which $4p^2 - u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 5 \times p^2$ by each of the 6 divisors $1, 2, 4, 5, 8$, and 10. We also note that each of the remaining divisors that $4p^2 - u$ can equal, is less than $4p^2$, giving a positive integer value for u.)
So then there are $24 - 6 = 18$ different integer values of $4p^2 - u$, and so there are exactly 18 different integer values of u for all prime numbers $p \geq 11$.

Therefore, $p = 7$ is the only prime number for which there are exactly 17 positive integer values of u for Koeller-rectangles with $n = 10$ and $r = \dfrac{u}{p^2}$.