Galois Contest

(Grade 10)

Wednesday, April 12, 2017
(in North America and South America)

Thursday, April 13, 2017
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes $\qquad$ ©2017 University of Waterloo

Do not open this booklet until instructed to do so.

Number of questions: 4 $\qquad$ **Each question is worth 10 marks**

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 💡

   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

2. **FULL SOLUTION** parts indicated by ✎

   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

   - Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
   - Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.

3. For questions marked 💡, place your answer in the appropriate box in the answer booklet and **show your work**.

4. For questions marked ✏️, provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.

5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

1. On Monday, Daniel had 90 cups, each of which was either purple or yellow. He distributed the cups among three boxes as follows:

   Box D: 9 purple and 23 yellow cups for a total of 32 cups
   Box E: 6 purple and 24 yellow cups for a total of 30 cups
   Box F: 28 cups in total

   (a) What percentage of the cups in Box E were purple?

   (b) Of the 90 cups that Daniel had on Monday, 30% were purple. How many of the cups in Box F were purple?

   (c) On Tuesday, Avril brought 9 more purple cups and included them with Daniel's cups. Barry brought some yellow cups and included them with Daniel's cups and Avril's cups. The percentage of cups that were purple was again 30%. How many cups did Barry bring?

2. The Breakfast Restaurant has a special pricing day. If a customer arrives at the restaurant between 4:30 a.m. and 7:00 a.m., the time that they arrive in hours and minutes becomes the price that they pay in dollars and cents. For example, if a customer arrives at 5:23 a.m., they will pay $5.23.

   (a) Abdi arrived at 5:02 a.m. and Caleigh arrived at 5:10 a.m. In total, how much did they pay?

   (b) Robert arrived 10 minutes before Emily, and both arrived during the period of the special pricing. In total, they paid $12.34. What were their arrival times?

   (c) Isaac and Jacob arrived together and Karla arrived after. All three arrived during the period of the special pricing. In total, they paid $18.55. What was the minimum amount that Karla could have paid?

   (d) Larry and Mio arrived separately during the period of the special pricing. In total, they paid $11.98. Determine the ranges of times during which Larry could have arrived.

3. A *tangent* to a circle is a line or line segment that touches the circle in exactly one place and would not touch the circle again, even if extended infinitely in both directions. When a tangent to a circle with centre O touches the circle at P, radius OP is perpendicular to the tangent.

(a) In the diagram, O is the centre of the circle with radius 18. QR is tangent to the circle at P. Line segment OQ intersects the circle at S. Determine the length of SQ.
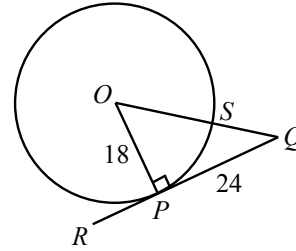
(b) A circle is said to be *inscribed* in a quadrilateral if each side of the quadrilateral is tangent to the circle. A circle with centre O is inscribed in quadrilateral $ABCD$, touching AB at E, BC at F, CD at G, and DA at H, as shown. If the radius of the circle is 12, $OB = 15$, $OC = 20$, and $\angle BAD = \angle ADC = 90°$, what is the perimeter of quadrilateral $ABCD$?
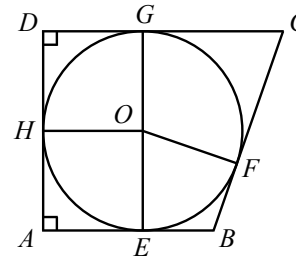
(c) Circles with centres O and C are inscribed in squares, as shown. The area of the larger square is 289 and the area of the smaller square is 49. If T, U and V lie on a straight line, determine the length of OC.
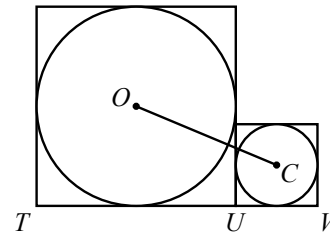
4. A *Koeller-rectangle*:

- is an m by n rectangle where m, n are integers with $m \geq 3$ and $n \geq 3$,
- has lines drawn parallel to its sides to divide it into 1 by 1 squares, and
- has the 1 by 1 squares along its sides unshaded and the 1 by 1 squares that do not touch its sides shaded.
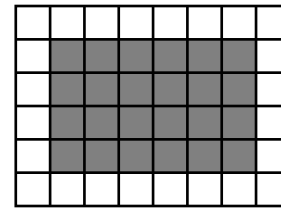
An example of a Koeller-rectangle with $m = 8$ and $n = 6$ is shown.

For a given Koeller-rectangle, let r be the ratio of the shaded area to the unshaded area.

(a) Determine the value of r for a Koeller-rectangle with $m = 14$ and $n = 10$.

(b) Determine all possible positive integer values of u for which there exists a Koeller-rectangle with $n = 4$ and $r = \dfrac{u}{77}$.

(c) Determine all prime numbers p for which there are exactly 17 positive integer values of u for Koeller-rectangles with $n = 10$ and $r = \dfrac{u}{p^2}$.

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Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2017.

Visit our website `cemc.uwaterloo.ca` to find

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- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

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